1. *(Brandon Jia)*: Let a and b be the real roots of the quadratic equation $x^2 - 98x + 1 = 0$. Compute the value of $\sqrt{a} + \sqrt{b}$.

   Answer: 10

   Solution: By Vieta's formulas, we have:

   $$a + b = 98$$

   $$ab = 1$$

   Note that a and b must be positive as both the sum and product are positive. Let $S = \sqrt{a}+\sqrt{b}$. Squaring both sides:

   $$S^2 = (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$$

   Substituting:

   $$S^2 = 98 + 2\sqrt{1} = 100$$

   Since a and b are positive, their square roots are positive, making $S > 0$. Thus:

   $$S = \sqrt{100} = \boxed{10}$$

2. *(Brandon Jia)*: Evaluate

$$\sum_{x=2}^{101}\sum_{y=2}^{101}\frac{1}{1+\log_y(x)\cdot\log_{xy}(x^{\log_y(xy)})}.$$

Answer: 5000

Solution: We first simplify the summand. Using logarithm rules,

$$\log_{xy}(x^{\log_y(xy)}) = \log_y(xy)\cdot\log_{xy}(x) = \frac{\ln(xy)}{\ln y}\cdot\frac{\ln x}{\ln(xy)} = \frac{\ln x}{\ln y} = \log_y(x)$$

This means that the entire denominator simplifies to:

$$1+\log_y(x)\cdot\log_y(x) = 1+(\log_y x)^2$$

Let the term in the summation be $f(x, y)$. The sum is:

$$\sum_{x=2}^{101}\sum_{y=2}^{101}\frac{1}{1+(\log_y x)^2}$$

Notice that swapping x and y gives:

$$f(y,x) = \frac{1}{1+(\log_x y)^2} = \frac{1}{1+\left(\frac{1}{\log_y x}\right)^2} = \frac{(\log_y x)^2}{(\log_y x)^2+1}$$

Adding symmetric pairs yields:

$$f(x,y)+f(y,x) = \frac{1}{1+(\log_y x)^2} + \frac{(\log_y x)^2}{(\log_y x)^2+1} = 1$$

The summation iterates through $x, y \in \{2, 3, \dots, 101\}$, providing $100 \times 100 = 10000$ total terms.

Notice that we can pair each (x, y) with (y, x). When $x = y$, $f(x, x) = \frac{1}{2}$, which matches the symmetric sum average. Thus, the total sum is half of the number of terms:

$$= \frac{10000}{2} = \boxed{5000}.$$

3. *(Brandon Jia)* Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(f(x)) = x^2 - 2$ for all real numbers x. Given that $f(3) = 5$, compute the value of $f(7)$.

Answer: 23

Solution 1: At $x = 3$:

$$f(f(3)) = 3^2 - 2 = 7$$

Since $f(3) = 5$:

$$f(5) = 7$$

Now, applying this again for $x = 5$:

$$f(f(5)) = 5^2 - 2 = 23$$

Since $f(5) = 7$, we have:

$$f(7) = \boxed{23}$$

Solution 2: Alternatively, we can manipulate the functional equation directly. Taking f of both sides yields:

$$f(f(f(x))) = f(x^2 - 2)$$

Substituting x with $f(x)$ yields:

$$f(f(f(x))) = f(x)^2 - 2$$

Thus:

$$f(x)^2 - 2 = f(x^2 - 2)$$

Therefore:

$$f(3)^2 - 2 = f(7) = \boxed{23}$$

4. *(Brandon Jia)* Let $f(x)$ be a monic polynomial of degree 12 whose roots are the integers x satisfying $-7 \le x \le -2$ or $2 \le x \le 7$. The value of

$$\left|\frac{f(i+1)}{f(i)}\right|^2$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: 38

Solution: Note that $f(x)$ is

$$f(x) = \prod_{k=2}^{7}(x^2 - k^2)$$

Because $f(x)$ has real coefficients, $\overline{f(z)} = f(\overline{z})$. We now evaluate the magnitude squared, for both the denominator and the numerator.

For $|f(i)|^2$:

$$f(i) = \prod_{k=2}^{7}(i^2 - k^2) = \prod_{k=2}^{7}(-1 - k^2) = (-1)^6 \prod_{k=2}^{7}(k^2 + 1)$$

Since $f(i)$ is entirely real, $|f(i)|^2 = (f(i))^2 = \prod_{k=2}^{7}(k^2+1)^2$.

For $|f(i+1)|^2$:

$$|f(i+1)|^2 = f(i+1)f(-i+1) = \prod_{k=2}^{7}\left((1+i)^2 - k^2\right)\left((1-i)^2 - k^2\right)$$

Since $(1 \pm i)^2 = \pm 2i$:

$$|f(i+1)|^2 = \prod_{k=2}^{7}(2i - k^2)(-2i - k^2) = \prod_{k=2}^{7}(k^4 - (2i)^2) = \prod_{k=2}^{7}(k^4 + 4)$$

By Sophie Germain's identity, $k^4 + 4 = ((k-1)^2+1)((k+1)^2+1)$.

Now, we have to evaluate the ratio. Let $P(m) = m^2 + 1$.

$$|f(i+1)|^2 = \prod_{k=2}^{7} P(k-1)P(k+1) = (P(1)P(2)\cdots P(6)) \times (P(3)P(4)\cdots P(8))$$

$$|f(i)|^2 = \prod_{k=2}^{7} P(k)^2 = (P(2)P(3)\cdots P(7)) \times (P(2)P(3)\cdots P(7))$$

We can telescope:

$$\left|\frac{f(i+1)}{f(i)}\right|^2 = \frac{P(1)}{P(7)} \times \frac{P(8)}{P(2)}$$

Substitute $P(m) = m^2 + 1$:

$P(1) = 2$, $P(2) = 5$, $P(7) = 50$, $P(8) = 65$.

$$\text{Ratio} = \frac{2 \times 65}{5 \times 50} = \frac{130}{250} = \frac{13}{25}$$

Thus, $m = 13$ and $n = 25$, making $m + n = \boxed{38}$.

5. *(Brandon Jia)* The sum

$$\sum_{k=3}^{2026} \frac{1}{(k-1)\sqrt{k-2}+(k-2)\sqrt{k-1}}$$

can be expressed as a fraction $\frac{m}{n}$ in lowest terms, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: 89

Solution: The idea is to rationalize each term so that the sum telescopes. Let $x = k-1$, so the summation bounds become $x = 2$ to $x = 2025$. The general term is:

$$\frac{1}{x\sqrt{x-1}+(x-1)\sqrt{x}}$$

We can now factor out the common term $\sqrt{x}\sqrt{x-1}$ from the denominator:

$$\frac{1}{\sqrt{x}\sqrt{x-1}(\sqrt{x}+\sqrt{x-1})}$$

We rationalize the denominator:

$$\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}\sqrt{x-1}(x-(x-1))} = \frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}\sqrt{x-1}} = \frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x}}$$

This telescopes:

$$\sum_{x=2}^{2025} \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x}}\right)$$

$$\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \cdots + \left(\frac{1}{\sqrt{2024}} - \frac{1}{\sqrt{2025}}\right)$$
$$= 1 - \frac{1}{\sqrt{2025}} = 1 - \frac{1}{45} = \frac{44}{45}$$

This gives $m = 44$, $n = 45$, giving $m+n = \boxed{89}$.

6. *(Brandon Jia)* Let x, y, and z be positive real numbers that satisfy

$$x + y + z = xyz = 2026.$$

Compute

$$\frac{(x+y)(y+z)(z+x)}{\sqrt{1+x^2}\sqrt{1+y^2}\sqrt{1+z^2}}.$$

Answer: 2026

Solution 1 (Trig Substitution): Because $x, y, z > 0$ and $x + y + z = xyz$, they correspond to the tangents of the angles of an acute triangle. Let $x = \tan A$, $y = \tan B$, and $z = \tan C$, where $A + B + C = \pi$. For the denominator, this leads to:

$$\sqrt{1+\tan^2 A}\sqrt{1+\tan^2 B}\sqrt{1+\tan^2 C} = \sec A \sec B \sec C = \frac{1}{\cos A \cos B \cos C}$$

For the numerator, using the tangent addition formula, $x + y = \tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$. Since $A + B = \pi - C$, we have $\sin(A + B) = \sin C$, meaning:

$$x + y = \frac{\sin C}{\cos A \cos B}$$

By symmetry, $(y + z) = \frac{\sin A}{\cos B \cos C}$ and $(z + x) = \frac{\sin B}{\cos C \cos A}$.

Multiplying them together yields:

$$(x+y)(y+z)(z+x) = \frac{\sin A \sin B \sin C}{\cos^2 A \cos^2 B \cos^2 C}$$

Evaluating the overall ratio:

$$\frac{\frac{\sin A \sin B \sin C}{\cos^2 A \cos^2 B \cos^2 C}}{\frac{1}{\cos A \cos B \cos C}} = \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} = \tan A \tan B \tan C = xyz$$

Since $xyz = 2026$, the expression evaluates to $\boxed{2026}$.

Solution 2 (Algebra): From the given equation $x + y + z = xyz$, we can rearrange it to isolate z:

$$z(xy - 1) = x + y \implies z = \frac{x+y}{xy-1}$$

Substituting into $1 + z^2$:

$$1 + z^2 = 1 + \left(\frac{x+y}{xy-1}\right)^2$$

$$1 + z^2 = \frac{(xy-1)^2 + (x+y)^2}{(xy-1)^2}$$

$$1 + z^2 = \frac{x^2y^2 - 2xy + 1 + x^2 + 2xy + y^2}{(xy-1)^2}$$

$$1 + z^2 = \frac{x^2y^2 + x^2 + y^2 + 1}{(xy-1)^2}$$

Factor the numerator by grouping:

$$1 + z^2 = \frac{(x^2+1)(y^2+1)}{(xy-1)^2}$$

Taking the square root of both sides (since all variables and $(xy - 1)$ are positive):

$$\sqrt{1+z^2} = \frac{\sqrt{1+x^2}\sqrt{1+y^2}}{xy-1}$$

Multiply both sides by $\sqrt{1+x^2}\sqrt{1+y^2}$ to find the denominator of our target expression:

$$\sqrt{1+x^2}\sqrt{1+y^2}\sqrt{1+z^2} = \frac{(1+x^2)(1+y^2)}{xy-1}$$

Now we rewrite the numerator $(x+y)(y+z)(z+x)$. We established above that $x+y = z(xy-1)$, and we do the same for the other two factors by substituting $z = \frac{x+y}{xy-1}$:

$$y + z = y + \frac{x+y}{xy-1} = \frac{xy^2 - y + x + y}{xy-1} = \frac{x(y^2+1)}{xy-1}$$

$$z + x = x + \frac{x+y}{xy-1} = \frac{x^2y - x + x + y}{xy-1} = \frac{y(x^2+1)}{xy-1}$$

Multiply these three factors together to get the full numerator:

$$(x+y)(y+z)(z+x) = (z(xy-1))\left(\frac{x(y^2+1)}{xy-1}\right)\left(\frac{y(x^2+1)}{xy-1}\right)$$

The $(xy - 1)$ terms partially cancel, grouping the variables together:

$$(x+y)(y+z)(z+x) = xyz\frac{(1+x^2)(1+y^2)}{xy-1}$$

Finally, divide the numerator by the denominator:

$$\frac{(x+y)(y+z)(z+x)}{\sqrt{1+x^2}\sqrt{1+y^2}\sqrt{1+z^2}} = \frac{xyz\frac{(1+x^2)(1+y^2)}{xy-1}}{\frac{(1+x^2)(1+y^2)}{xy-1}}$$

This reduces to xyz, which is equal to 2026. Thus, the expression evaluates to $\boxed{2026}$.

7. *(Christopher Liang)* Determine the sum of all possible values of $x + y$, where x and y are positive integers satisfying the equation

$$x^3 - y^3 = 13(x^2 + y^2).$$

Answer: 20

Solution: We first factor the left side:

$$(x - y)(x^2 + xy + y^2) = 13(x^2 + y^2)$$

Let $x - y = k$. Since x, y are positive integers and $x^3 - y^3 > 0$, it follows that $x > y$ and that k is a positive integer. We now substitute $x - y = k$:

$$k(x^2 + xy + y^2) = 13(x^2 + y^2)$$

$$k(x^2 + y^2) + kxy = 13(x^2 + y^2) \implies kxy = (13 - k)(x^2 + y^2)$$

Since $x, y > 0$ we know that $x^2 + y^2 > 0$ and $xy > 0$, forcing $13 - k > 0$, meaning $k \leq 12$. By AM-GM, $x^2 + y^2 \geq 2xy$. Substituting this bound:

$$kxy \geq (13 - k)(2xy) \implies k \geq 26 - 2k \implies 3k \geq 26 \implies k \geq 9$$

Thus, k is either 9, 10, 11, or 12.

- If $k = 10$:

  $$10xy = 3(x^2 + y^2)$$

  Since $x^2 + y^2 = (x - y)^2 + 2xy = 100 + 2xy$:

  $$10xy = 3(100 + 2xy) \implies 10xy = 300 + 6xy \implies 4xy = 300 \implies xy = 75$$

  We have a system: $x - y = 10$ and $xy = 75$. Substituting $x = y + 10$:

  $$y(y + 10) = 75 \implies y^2 + 10y - 75 = 0 \implies (y + 15)(y - 5) = 0$$

  Since $y > 0$, $y = 5$. This gives $x = 15$.

- If $k = 9$: $9xy = 4(81 + 2xy) \implies xy = 324$. $y(y + 9) = 324 \implies y^2 + 9y - 324 = 0$. The roots are irrational.

- If $k = 11$: $11xy = 2(121 + 2xy) \implies 7xy = 242 \implies xy = 242/7$, not an integer.

- If $k = 12$: $12xy = 1(144 + 2xy) \implies 10xy = 144$, not an integer.

Thus, there is only one integer solution pair, so the sum of all possible values is $15 + 5 = \boxed{20}$.

8. *(Brandon Jia)* Let a, b, and c be the complex roots of the cubic equation $x^3 - x^2 - x - 1 = 0$. Compute the value of

$$\left(\frac{1+a}{1-a}\right)^4 + \left(\frac{1+b}{1-b}\right)^4 + \left(\frac{1+c}{1-c}\right)^4.$$

Answer: 131

Solution: Let $y = \frac{1+x}{1-x}$. Solving for x in terms of y:

$$y(1-x) = 1+x \implies y - yx = 1 + x \implies x(y+1) = y - 1 \implies x = \frac{y-1}{y+1}$$

Substitute x into the original cubic:

$$\left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right)^2 - \left(\frac{y-1}{y+1}\right) - 1 = 0$$

Multiply the entire equation by $(y+1)^3$ to clear the denominators:

$$(y-1)^3 - (y-1)^2(y+1) - (y-1)(y+1)^2 - (y+1)^3 = 0$$

After simplification, this simplifies to:

$$y^3 + 3y^2 - y + 1 = 0$$

The values $\frac{1+a}{1-a}$, $\frac{1+b}{1-b}$, and $\frac{1+c}{1-c}$ are the roots y_1, y_2, y_3 of this new polynomial. We must find $S_4 = y_1^4 + y_2^4 + y_3^4$. We define the power sums $S_n = y_1^n + y_2^n + y_3^n$ and use Newton's Sums (alternatively, one could also use Vieta's formulas):

- $S_1 = -3$
- $S_2 = S_1^2 - 2(y_1y_2 + y_2y_3 + y_3y_1) = (-3)^2 - 2(-1) = 11$
- $S_3 = -3S_2 + S_1 - 3(1) = -3(11) + (-3) - 3 = -39$
- $S_4 = -3S_3 + S_2 - S_1 = -3(-39) + 11 - (-3) = 117 + 11 + 3 = \boxed{131}$.

9. *(Christopher Liang)* For real numbers a, b, c, and d, the following equation holds:

$$\frac{3d+9}{a+b+c} = \frac{3c+16}{a+b+d} = \frac{3b-24}{a+c+d} = \frac{3a-1}{b+c+d}$$

Determine $\frac{b-c}{a-d}$, given that $a \neq d$.

Answer: 4

Solution: Let the common value of all four of the equal fractions be k. Add k times the missing variable to both sides of each equation to complete a sum:

$$3d+9 = k(a+b+c) \implies d(k+3)+9 = kS$$

$$3c+16 = k(a+b+d) \implies c(k+3)+16 = kS$$

$$3b-24 = k(a+c+d) \implies b(k+3)-24 = kS$$

$$3a-1 = k(b+c+d) \implies a(k+3)-1 = kS$$

Subtract the c-equation from the b-equation:

$$(b-c)(k+3)-40 = 0 \implies (b-c)(k+3) = 40$$

Subtract the d-equation from the a-equation:

$$(a-d)(k+3)-10 = 0 \implies (a-d)(k+3) = 10$$

Divide the first result by the second to find the requested ratio:

$$\frac{(b-c)(k+3)}{(a-d)(k+3)} = \frac{40}{10} \implies \frac{b-c}{a-d} = \boxed{4}$$

10. *(Christopher Liang)* Suppose a, b, and c are positive real numbers satisfying

$$\begin{aligned}&(12\arctan(a)+8\arctan(3a))^4+\left(4\arcsin\left(\frac{b}{\sqrt{c}}\right)\right)^4\\&+\left(\log_b\left(\left(\frac{c}{\sqrt{33}}\right)^{2\pi}\right)\right)^4+4096\left(4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right)\right)^4\\&=8\pi\left(12\arctan(a)+8\arctan(3a)\right)\left(\log_b\left(\left(\frac{c}{\sqrt{33}}\right)^{2\pi}\right)\right)\left(4\arcsin\left(\frac{b}{\sqrt{c}}\right)\right).\end{aligned}$$

The maximum value of $(ab)^2$ can be expressed in the form $n-\sqrt{m}$, where m is a positive integer that is not necessarily squarefree. Find $n+m$.

Answer: 207

Solution: We introduce four variables to organize the equation:

$$X=12\arctan(a)+8\arctan(3a)$$

$$Y=\log_b\left(\left(\frac{c}{\sqrt{33}}\right)^{2\pi}\right)$$

$$Z=4\arcsin\left(\frac{b}{\sqrt{c}}\right)$$

$$W=4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right)$$

It is well known that $W=\frac{\pi}{4}$ (see Machin's Formula for derivation). Thus, the constant term $4096W^4=4096\left(\frac{\pi}{4}\right)^4=16\pi^4=(2\pi)^4$.

The equation becomes:

$$X^4+Y^4+Z^4+(2\pi)^4=8\pi XYZ$$

Applying the AM-GM, $A^4+B^4+C^4+D^4\geq 4\sqrt[4]{A^4B^4C^4D^4}$:

$$X^4+Y^4+Z^4+(2\pi)^4\geq 4|XYZ(2\pi)|=8\pi|XYZ|\geq 8\pi XYZ$$

Note that this is the equality case for AM-GM, so $X=Y=Z=2\pi$ (note that they must be positive as $a,b,c>0$).

First, we evaluate Z.

$$4\arcsin\left(\frac{b}{\sqrt{c}}\right)=2\pi\implies\arcsin\left(\frac{b}{\sqrt{c}}\right)=\frac{\pi}{2}\implies c=b^2$$

Now, Y:

$$\log_b\left(\left(\frac{c}{\sqrt{33}}\right)^{2\pi}\right)=2\pi\implies\log_b\left(\frac{c}{\sqrt{33}}\right)=1\implies c=b\sqrt{33}$$

Setting the two above gives $b^2=33$.

For X:

$$12\arctan(a)+8\arctan(3a)=2\pi\implies 3\arctan(a)+2\arctan(3a)=\frac{\pi}{2}$$

Let $\alpha=\arctan(a)$ and $\beta=\arctan(3a)$. Then $3\alpha+2\beta=\frac{\pi}{2}\implies 2\beta=\frac{\pi}{2}-3\alpha$.

Taking the tangent of both sides:

$$\tan(2\beta) = \cot(3\alpha)$$

Using $\tan(2\beta) = \frac{6a}{1-9a^2}$ and $\tan(3\alpha) = \frac{3a-a^3}{1-3a^2}$:

$$\frac{6a}{1-9a^2} = \frac{1-3a^2}{3a-a^3} \implies 18a^2 - 6a^4 = 1 - 12a^2 + 27a^4$$

$$33a^4 - 30a^2 + 1 = 0$$

Using the quadratic formula to solve for a^2:

$$a^2 = \frac{30 \pm \sqrt{900-132}}{66} = \frac{15 \pm \sqrt{192}}{33}$$

The "+" root corresponds to the sum angle matching $3\pi/2$, but $3\alpha + 2\beta = \pi/2$. Therefore, $a^2 = \frac{15-\sqrt{192}}{33}$.

We can now multiply our terms:

$$(ab)^2 = a^2b^2 = \left(\frac{15-\sqrt{192}}{33}\right)(33) = 15 - \sqrt{192}$$

The problem requests $n+m$ given the format $n - \sqrt{m}$. Thus, $n = 15$ and $m = 192$.

$$15 + 192 = \boxed{207}$$