Integrated Mathematics Tournament

Combinatorics Round Problems 2026 IMT

1. A subset $A \subseteq \{1, 2, 3, \ldots, 12\}$ is called *peaceful* if no two elements of A are consecutive. Determine the number of nonempty peaceful subsets.

2. A word is formed using exactly two I's, three M's, and three T's. Determine the number of such words in which no two T's are adjacent.

3. A lattice path from $(0, 0)$ to $(6, 6)$ consists of 6 steps right and 6 steps up. A path is called *balanced* if it never goes above the line $y = x$, and it touches the line $y = x$ only at its starting and ending points. Determine the number of balanced paths.

4. Let $\mathcal{S}$ be the set of all 10! permutations of the sequence $(1, 2, 3, \ldots, 10)$. An adjacent swap of two elements a_i and a_{i+1} in a permutation is called permissible if the sum $a_i + a_{i+1}$ is odd. Let $\mathcal{R} \subset \mathcal{S}$ be the set of all permutations that can be obtained from the initial identity permutation $(1, 2, 3, \ldots, 10)$ through a finite sequence of permissible adjacent swaps. Compute the number of elements in $\mathcal{R}$.

5. A single-elimination tournament is organized for 2^n players, where n is a positive integer. The players are randomly assigned to the 2^n available slots of a standard tournament bracket. In every match, each participant has a 50% probability of winning and advancing to the next round. In terms of n, the probability that Player 1 and Player 2 compete in a match against one another at any point during the tournament is a^{b-n} where a, b are positive integers. Compute $a + b$.

6. A 4×4 grid of unit squares is initially colored entirely white. An operation consists of choosing any 2×2 subgrid and toggling the colors of its four unit squares (changing white squares to black and black squares to white). Determine the total number of distinct color patterns of the 4×4 grid that can be achieved through any sequence of such operations.

7. There are 2026 lamps arranged in a circle, each either on or off. A coloring is called *mysterious* if every block of 1013 consecutive lamps contains an odd number of lamps that are on.

   Two mysterious colorings are considered the same if one can be obtained from the other by rotating the circle. The number of distinct mysterious colorings can be written as $\frac{2^n+n}{m}$. find $n + m$.

8. A committee consists of 20 members, each of whom is either a knight, who always tells the truth, or a knave, who always lies. Each member is assigned a unique ID number from the set $\{1, 2, \ldots, 20\}$. Every member is asked the same question: "Is the number of knights in the committee strictly greater than my ID number?" Let k be the total number of members who answer "Yes." Determine the number of possible values of k.

9. Let n be a positive integer and let $S = \{1, 2, 3, \ldots, 2n\}$. A non-empty subset $A \subseteq S$ is said to be consistent if the parity of the maximum element of A is equal to the parity of the sum of the elements of A. What is the number of consistent subsets of S if n=6?

10. A galactic summit has $N = 2026$ delegates in attendance. Every delegate has the technology to establish a direct, two-way communication link with any other delegate. However, each delegate has exactly one bitter rival among the other 2025 delegates (the rivalry is mutual). No delegate will ever establish a link with their rival.

The summit organizers want to activate a specific subset of the remaining possible communication links to form a connected network that bridges all 2026 delegates. Furthermore, to save power, this network must be a *spanning tree* (meaning there are exactly 2025 active links and no closed loops).

The total number of valid spanning trees that can be formed can be uniquely written in the form:

$$N^A \cdot (N-2)^B$$

where A and B are positive integers. Compute $A + B$.