Combinatorics Round Solutions 2026 IMT

1. *(Brandon Jia)*: A subset $A \subseteq \{1, 2, 3, \ldots, 12\}$ is called *peaceful* if no two elements of A are consecutive. Determine the number of nonempty peaceful subsets.

   Answer: 376

   Solution: Let F_n be the number of peaceful subsets of $\{1, 2, \ldots, n\}$, including the empty set. We build up from small values:

   - For $n = 1$, the subsets are $\emptyset$ and $\{1\}$. Total $= 2$.
   - For $n = 2$, the subsets are $\emptyset$, $\{1\}$, and $\{2\}$. Total $= 3$.
   - For $n = 3$, the subsets are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$. Total $= 5$.

   These counts $2, 3, 5, \ldots$ are Fibonacci numbers: the total number of peaceful subsets of a set of size n is F_{n+2} (where $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \ldots$). Indeed, a peaceful subset of $\{1, \ldots, n\}$ either omits n (giving F_{n+1} subsets of $\{1, \ldots, n-1\}$) or contains n and hence omits $n-1$ (giving F_n subsets of $\{1, \ldots, n-2\}$), so the counts satisfy the Fibonacci recurrence. We want the value for $n = 12$, which corresponds to F_{14}.

   Listing the Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377$. So, there are 377 total peaceful subsets. We subtract the 1 empty set to get $377 - 1 = \boxed{376}$ nonempty peaceful subsets.

2. *(Christopher Liang)*: A word is formed using exactly two I's, three M's, and three T's. Determine the number of such words in which no two T's are adjacent.

Answer: 200

Solution: We build the word in two stages so that no two T's can be adjacent. First, we arrange the non-T letters (the two I's and three M's) to form a barrier. The number of ways to arrange these 5 letters is:

$$\frac{5!}{2!\,3!} = \frac{120}{12} = 10$$

Placed in a row, these 5 letters create exactly 6 slots where a T can go: one before the first letter, four between letters, and one after the last.

To ensure that no two T's are adjacent, we must select exactly 3 of these 6 slots to place our three identical T's. The number of ways to do this is:

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

Since each arrangement of the barrier letters is independent of the slot selection for the T's, we multiply the possibilities together. Total valid words $= 10 \times 20 = \boxed{200}$.

3. *(Christopher Liang)*: A lattice path from $(0,0)$ to $(6,6)$ consists of 6 steps right and 6 steps up. A path is called *balanced* if it never goes above the line $y = x$, and it touches the line $y = x$ only at its starting and ending points. Determine the number of balanced paths.

Answer: 42

Solution 1: Notice that since the path can only touch $y = x$ at the very beginning and the very end, its first move *must* be to the right (to $(1,0)$) and its very last move *must* be up (from $(6,5)$ to $(6,6)$).

If we strip away this fixed first and last step, we are left looking at a path that travels from $(1,0)$ to $(6,5)$. This intermediate path consists of 5 right steps and 5 up steps. Furthermore, since the original path was strictly below $y = x$ between the endpoints, this new shifted path must not cross the line $y = x - 1$.

If we shift our coordinate system by treating $(1,0)$ as our new origin $(0,0)$, our destination becomes $(5,5)$, and the bounding boundary simply becomes the standard diagonal $y = x$. The number of paths from $(0,0)$ to (n,n) that do not cross above $y = x$ is a classic combinatorial sequence: the Catalan numbers, given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

We just need the 5th Catalan number:

$$C_5 = \frac{1}{6}\binom{10}{5} = \frac{1}{6}(252) = \boxed{42}$$

Solution 2: Because the path can only touch the line $y = x$ at the very beginning and the very end, its first step must strictly be to the right (from $(0,0)$ to $(1,0)$), and its final step must strictly be up (from $(6,5)$ to $(6,6)$).

The problem then reduces to finding the number of valid intermediate paths from $(1,0)$ to $(6,5)$ that never touch or cross the line $y = x$.

First, we find the total number of unrestricted paths from $(1,0)$ to $(6,5)$. This requires $6 - 1 = 5$ steps right and $5 - 0 = 5$ steps up, giving a total of 10 steps. The number of such paths is:

$$\binom{10}{5} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$$

Next, we count the "bad" paths—those that touch the line $y = x$ at some point. Consider any bad path. At the exact moment it first touches the line $y = x$, we reflect the entire remaining portion of the path across the line $y = x$. By doing this, every subsequent "up" step becomes a "right" step, and every "right" step becomes an "up" step.

Because of this geometric reflection, the original destination point of $(6,5)$ is reflected across the line $y = x$ to a new destination of $(5,6)$. Thus, every bad path from $(1,0)$ to $(6,5)$ forms a one-to-one bijection with a path from $(1,0)$ to $(5,6)$.

The number of paths from $(1,0)$ to $(5,6)$ requires $5 - 1 = 4$ steps right and $6 - 0 = 6$ steps up. The number of such paths is:

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

To find the number of valid, strictly balanced paths, we subtract these bad paths from the total unrestricted paths:

$$252 - 210 = \boxed{42}$$

4. *(Brandon Jia)*: Let $\mathcal{S}$ be the set of all 10! permutations of the sequence $(1, 2, 3, \ldots, 10)$. An adjacent swap of two elements a_i and a_{i+1} in a permutation is called permissible if the sum $a_i + a_{i+1}$ is odd. Let $\mathcal{R} \subset \mathcal{S}$ be the set of all permutations that can be obtained from the initial identity permutation $(1, 2, 3, \ldots, 10)$ through a finite sequence of permissible adjacent swaps. Compute the number of elements in $\mathcal{R}$.

Answer: 252

Solution: Consider what the permissible-swap condition restricts. For a sum to be odd, the two numbers must have opposite parities: one even and one odd.

This implies we can *only* swap an even number with an odd number. Because of this, the relative order of the even numbers $(2, 4, 6, 8, 10)$ can never be changed. Similarly, the relative order of the odd numbers $(1, 3, 5, 7, 9)$ is strictly locked in place.

However, since an even and an odd may be swapped freely, we can interleave the two fixed sequences in any order. The only constraint is that the internal order of the five evens and the internal order of the five odds are each preserved.

The number of valid permutations is simply the number of ways to choose the 5 positions out of 10 that will be occupied by the even numbers (the remaining 5 positions will naturally be filled by the odd numbers).

$$\binom{10}{5} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = \boxed{252}$$

Note: This problem was partially inspired by 2026 USAJMO Problem 2.

5. *(Brandon Jia)*: A single-elimination tournament is organized for 2^n players, where n is a positive integer. The players are randomly assigned to the 2^n available slots of a standard tournament bracket. In every match, each participant has a 50% probability of winning and advancing to the next round. In terms of n, the probability that Player 1 and Player 2 compete in a match against one another at any point during the tournament is a^{b-n} where a, b are positive integers. Compute $a + b$.

Answer: 3

Solution: Player 1 and Player 2 play a role symmetric to that of any other pair of players in the tournament.

First, we determine how many pairs of players are in the tournament:

$$\text{Total Pairs} = \binom{2^n}{2} = \frac{2^n(2^n - 1)}{2} = 2^{n-1}(2^n - 1)$$

Next, we count how many matches take place. In a single-elimination tournament with 2^n players, exactly one player goes undefeated and every other player loses exactly one match, so there are exactly $2^n - 1$ matches in total.

Due to the initial random assignment and the 50/50 win chance for every single match, every possible matchup is equally likely to occur in the tournament. Since exactly $2^n - 1$ unique pairings will face off against each other out of the total pool of pairs, the probability that the specific pairing of (Player 1, Player 2) is one of the matches is simply:

$$\text{Probability} = \frac{\text{Number of Matches}}{\text{Total Pairs}} = \frac{2^n - 1}{2^{n-1}(2^n - 1)} = \frac{1}{2^{n-1}} = 2^{1-n}$$

The requested answer extraction is $2 + 1 = \boxed{3}$.

6. *(Christopher Liang)*: A 4×4 grid of unit squares is initially colored entirely white. An operation consists of choosing any 2×2 subgrid and toggling the colors of its four unit squares (changing white squares to black and black squares to white). Determine the total number of distinct color patterns of the 4×4 grid that can be achieved through any sequence of such operations.

Answer: 512

Solution 1 (Linear Algebra): We treat this as a linear algebra problem over the field of two elements $\mathbb{F}_2$. The grid has 16 squares, so any coloring can be represented as a vector in $\mathbb{F}_2^{16}$.

There are exactly 9 possible 2×2 subgrids we can choose to toggle (anchored by their top-left corners from row 1 to 3, and column 1 to 3). Toggling a subgrid corresponds to adding a specific vector to our grid's current state. Since toggling the same subgrid twice brings those squares back to their original state, each of the 9 subgrids should be toggled either 0 or 1 times.

This means there are $2^9 = 512$ combinations of operations we can perform.

It remains to determine whether any two of these combinations yield the same pattern, which is equivalent to asking whether the 9 operation vectors are linearly independent.

Assume there is a subset of operations that leaves the entire grid entirely white (a zero vector).

- Look at the top-left corner $(1, 1)$. It is *only* affected by the top-left 2×2 subgrid. For $(1, 1)$ to remain white, this top-left operation must not be used.
- By the exact same logic on the other three corners, the top-right, bottom-left, and bottom-right operations must also not be used.
- Now look at the edge square $(1, 2)$. It is affected by the top-left operation and the top-middle operation. Since we already proved the top-left operation is off, the top-middle operation must also be off to keep $(1, 2)$ white.
- Continuing this argument forces all 9 operations to be 0.

Since the only way to get a blank grid is to do nothing, the 9 vectors are linearly independent. Therefore all 2^9 combinations produce distinct patterns. The total number of patterns is $2^9 = \boxed{512}$.

Solution 2: We can begin the same way from Solution 1 to find that there are 512 combinations of operations that we can perform. Again, we have to figure out:

Does every single one of these 512 combinations create a unique pattern? To prove they do, we just need to prove that the *only* way to get a completely blank (all white) grid is to do absolutely nothing (0 toggles). If doing nothing is the only way to get a blank grid, then no two different combinations can cancel each other out to produce the exact same pattern.

Assume we have performed a sequence of operations that leaves the entire grid completely white, and examine the squares systematically:

- The top-left corner $(1, 1)$: The *only* 2×2 subgrid that covers this square is the top-left subgrid. If the square $(1, 1)$ ends up white, we must not have used the top-left subgrid. It is strictly "off".
- The square to its right $(1, 2)$: This square is only affected by the top-left subgrid and the top-middle subgrid. Since we just established the top-left subgrid is "off", the top-middle subgrid must also be "off" in order for $(1, 2)$ to remain white.

- The top-right corner $(1, 4)$: Similar to the first step, this square is only affected by the top-right subgrid. For it to remain white, the top-right subgrid must be "off".

At this point, we know that none of the subgrids in the top row were used.

Now we move to the second row and apply the same logic:

- The left-edge square $(2, 1)$: This is affected by the top-left subgrid and the middle-left subgrid. Since the top row is off, the middle-left subgrid must also be off to keep this square white.
- The inner square $(2, 2)$: This is affected by four subgrids. However, we already know the top-left, top-middle, and middle-left are all off. Thus, the exact center subgrid must also be off to keep $(2, 2)$ white.

Continuing from top to bottom and left to right, the same argument forces every subgrid to be off in order to keep the grid white.

Because the only way to create a blank grid is to do absolutely nothing, no two combinations of moves overlap. Thus, all 2^9 combinations create distinctly different patterns. The total number of patterns is $2^9 = \boxed{512}$.

7. *(Christopher Liang)*: There are 2026 lamps arranged in a circle, each either on or off. A coloring is called *mysterious* if every block of 1013 consecutive lamps contains an odd number of lamps that are on.

Two mysterious colorings are considered the same if one can be obtained from the other by rotating the circle. The number of distinct mysterious colorings can be written as $\frac{2^n+n}{m}$. find $n+m$.

Answer: 2025

Solution: We translate the problem into algebra. Let the lamps be $x_1, x_2, \ldots, x_{2026} \in \{0,1\}$. For any index i, the block of 1013 lamps starting at i must sum to an odd number:

$$\sum_{j=i}^{i+1012} x_j \equiv 1 \pmod 2$$

If we shift our window by one position, the sum must remain odd:

$$\sum_{j=i+1}^{i+1013} x_j \equiv 1 \pmod 2$$

Subtracting the first equation from the second leaves us with $x_{i+1013} - x_i \equiv 0 \pmod 2$, which implies $x_{i+1013} = x_i$. Because 1013 is half of 2026, this means every lamp must be exactly the same state as the lamp diametrically opposite it. Thus, the second half of the circle is just an identical repeating clone of the first 1013 lamps. Because of this repetition, *any* sequence of 1013 consecutive lamps around the circle is simply a cyclic permutation of the first 1013 lamps, and therefore contains the exact same number of ON lamps.

Therefore, a coloring is valid if and only if the first 1013 lamps contain an odd number of ON lamps. Out of the 2^{1013} possible half-circle sequences, exactly half of them have an odd sum, giving us 2^{1012} valid sequences.

Now, we must account for rotational symmetry using Burnside's Lemma over the cyclic group C_{1013}. Number of orbits $= \frac{1}{1013}\sum_{d\in C_{1013}}$(number of valid sequences fixed by rotation d).

The only divisors/rotations to check are the identity (0 shifts) and the 1012 non-trivial shifts as 1013 is prime.

- Identity (0 shifts): Fixes all 2^{1012} valid sequences.
- Shift by 1 (and all other 1011 shifts): For a sequence to be fixed by a shift of 1, every lamp must be identical. Since the sum must be odd, they must all be 1s (ON). The sum of 1013 ones is 1013, which is indeed odd. So exactly 1 sequence is fixed by these shifts.

Applying the lemma:

$$\text{Distinct colorings} = \frac{1}{1013}\left(2^{1012} + 1012 \times 1\right) = \frac{2^{1012} + 1012}{1013}$$

This matches the requested format $\frac{2^n+n}{m}$ with $n = 1012$ and $m = 1013$. The sum is $1012 + 1013 = \boxed{2025}$.

8. *(Brandon Jia)*: A committee consists of 20 members, each of whom is either a knight, who always tells the truth, or a knave, who always lies. Each member is assigned a unique ID number from the set $\{1, 2, \ldots, 20\}$. Every member is asked the same question: ”Is the number of knights in the committee strictly greater than my ID number?” Let k be the total number of members who answer ”Yes.” Determine the number of possible values of k.

Answer: 11

Solution: We classify the respondents by their truth-telling nature. Let N be the total number of knights in the committee ($0 \leq N \leq 20$). The question asked of member i is: “Is $N > i$?”.

- A knight tells the truth, so they answer ”Yes” if $i < N$.
- A knave lies. The truth is $i \geq N$, so they will falsely answer ”Yes” if $i \geq N$.

We count k, the total number of “Yes” answers. First consider the edge case $N = 0$. If there are 0 knights, then all 20 members are knaves. The statement “$0 > i$” is false for every ID, and since knaves lie, all 20 members answer “Yes”. Hence $k = 20$ is possible.

Now assume $N \geq 1$. There are exactly $N - 1$ available IDs strictly less than N (from 1 to $N - 1$). Let x be the number of knights holding these IDs. Since they are knights and $i < N$, all x of them will answer ”Yes”. The remaining $N - x$ knights must hold IDs $\geq N$. There are a total of $20 - N + 1 = 21 - N$ IDs available in the $\geq N$ range. The number of knaves holding these $\geq N$ IDs is therefore $(21 - N) - (N - x) = 21 - 2N + x$. All of these knaves will answer ”Yes”.

The total number of ”Yes” answers is $k = x + (21 - 2N + x) = 21 - 2(N - x)$. Let $m = N - x$, which represents the number of knights with IDs $\geq N$.

Since $x \geq 0$, we have $m \leq N$. Since $x \leq N - 1$ (there are only $N - 1$ IDs below N), we have $m = N - x \geq 1$. Furthermore, m cannot exceed the total number of available IDs $\geq N$, so $m \leq 21 - N$. Thus, $1 \leq m \leq \min(N, 21 - N)$.

By choosing an optimal N (like 10 or 11), the maximum possible value for $\min(N, 21 - N)$ is 10. Thus, m can be any integer from 1 to 10. Plugging this into our equation $k = 21 - 2m$, we find that k can be any odd number from $21 - 2(10) = 1$ up to $21 - 2(1) = 19$. This gives us 10 possible odd values. Including the $k = 20$ we found earlier, there are exactly $10 + 1 = \boxed{11}$ possible values for k.

9. *(Brandon Jia)*: Let n be a positive integer and let $S = \{1, 2, 3, \ldots, 2n\}$. A non-empty subset $A \subseteq S$ is said to be consistent if the parity of the maximum element of A is equal to the parity of the sum of the elements of A. What is the number of consistent subsets of S if $n = 6$?

Answer: 2048

Solution: We will fix a maximum element.

For any fixed maximum element $m \in S$ such that $m > 1$, consider the set of all subsets of S whose maximum element is exactly m. There are 2^{m-1} such subsets, as each element in $\{1, 2, \ldots, m-1\}$ may either be included in or excluded from the subset.

We can pair these 2^{m-1} subsets by the "toggling" of the element 1: if a subset A contains 1, we pair it with $A \setminus \{1\}$; if it does not, we pair it with $A \cup \{1\}$. Because $m > 1$, this operation does not change the maximum element of the subset. However, adding or removing 1 flips the parity of the subset's sum.

Consequently, in every such pair, exactly one subset will have a sum whose parity matches the parity of m, and exactly one will not. This implies that for every $m \in \{2, 3, \ldots, 2n\}$, there are exactly 2^{m-2} consistent subsets.

We must handle the edge case where $m = 1$ separately. The only subset with a maximum element of 1 is $\{1\}$ itself. Its maximum is 1 (odd) and its sum is 1 (odd), meaning this 1 subset is consistent.

Given $n = 6$, our set is $S = \{1, 2, \ldots, 12\}$. Summing the counts for all possible maximum elements m up to 12, the total number of consistent subsets is:

$$1 + \sum_{m=2}^{12} 2^{m-2} = 1 + (2^0 + 2^1 + 2^2 + \cdots + 2^{10})$$

By summing the resulting geometric series, we arrive at:

$$1 + (2^{11} - 1) = 2^{11} = \boxed{2048}$$

10. *(Brandon Jia)*: A galactic summit has $N = 2026$ delegates in attendance. Every delegate has the technology to establish a direct, two-way communication link with any other delegate. However, each delegate has exactly one bitter rival among the other 2025 delegates (the rivalry is mutual). No delegate will ever establish a link with their rival.

The summit organizers want to activate a specific subset of the remaining possible communication links to form a connected network that bridges all 2026 delegates. Furthermore, to save power, this network must be a *spanning tree* (meaning there are exactly 2025 active links and no closed loops).

The total number of valid spanning trees that can be formed can be uniquely written in the form:

$$N^A \cdot (N-2)^B$$

where A and B are positive integers. Compute $A + B$.

Answer: 2024

Solution 1 (Matrix–Tree Theorem): We must count the spanning trees of the complete graph K_N with a perfect matching M (the rivalries) removed. Call this graph $G = K_N \setminus M$.

The number of spanning trees is related to the eigenvalues of the graph's Laplacian matrix. The Laplacian of our graph G is simply $L_G = L_{K_N} - L_M$.

- For the complete graph K_N, the eigenvalues are 0 (multiplicity 1) and N (multiplicity $N-1$).
- The perfect matching M consists of $N/2$ disjoint edges. The Laplacian of a single edge has eigenvalues 0 and 2. Thus, L_M has eigenvalues 0 (multiplicity $N/2$) and 2 (multiplicity $N/2$).

Because M is a regular graph and K_N is highly symmetric, L_M and L_{K_N} share a common eigenbasis, so we may subtract the eigenvalues of L_M from those of L_{K_N} eigenvector by eigenvector. The all-ones vector accounts for the eigenvalue 0 in both, giving $0 - 0 = 0$ for L_G (as expected for a connected graph). For the remaining $N-1$ eigenvectors (which correspond to the eigenvalue N in K_N), L_M subtracts 2 exactly $N/2$ times, and subtracts 0 the remaining $N/2 - 1$ times.

Thus, the non-zero eigenvalues of L_G are:

- $(N-2)$ with multiplicity $N/2$
- $(N-0) = N$ with multiplicity $N/2 - 1$

According to the Matrix Tree Theorem, the number of spanning trees is $\frac{1}{N}$ times the product of the non-zero eigenvalues:

$$\text{Spanning Trees} = \frac{1}{N}\left[(N-2)^{N/2} \cdot N^{N/2-1}\right] = N^{N/2-2} \cdot (N-2)^{N/2}$$

This is the requested format $N^A \cdot (N-2)^B$, with $A = N/2 - 2$ and $B = N/2$. Therefore

$$A + B = (N/2 - 2) + N/2 = N - 2,$$

and substituting $N = 2026$ gives $2026 - 2 = \boxed{2024}$.

Solution 2 (Inclusion–Exclusion): Since we are counting spanning trees while forbidding a specific set of edges, we use the Principle of Inclusion–Exclusion (PIE) together with Cayley's formula. Recall that the complete graph K_N has exactly N^{N-2} spanning trees.

$$\text{Valid Trees} = \sum_{j=0}^{N/2} (-1)^j \times (\text{Number of trees containing } j \text{ specific forbidden edges})$$

We need the number of spanning trees of K_N that contain a prescribed set of edges. There is a standard extension of Cayley's formula for this: if the prescribed edges form a forest whose k components have sizes $c_1, c_2, \ldots, c_k$, then the number of spanning trees of K_N containing all of them is:

$$N^{k-2} \cdot c_1 \cdot c_2 \cdots c_k.$$

Taking no forced edges ($k = N$ with every $c_i = 1$) recovers the count N^{N-2} above. If we select j edges from the matching M, they are pairwise disjoint, so they form j components of size 2; the remaining $N - 2j$ vertices are isolated components of size 1. The total number of components is $k = j + (N-2j) = N - j$. The product of the component sizes is $2^j \cdot 1^{N-2j} = 2^j$. Substituting into the formula above, the number of spanning trees containing these j edges is:

$$N^{(N-j)-2} \cdot 2^j = N^{N-j-2} \cdot 2^j$$

There are $\binom{N/2}{j}$ ways to choose which j matching edges to force into the tree, so inclusion–exclusion gives:

$$\text{Total Valid Trees} = \sum_{j=0}^{N/2} (-1)^j \binom{N/2}{j} \left(N^{N-j-2} \cdot 2^j\right)$$

Factoring out N^{N-2}:

$$\text{Total Valid Trees} = N^{N-2} \sum_{j=0}^{N/2} \binom{N/2}{j} (-1)^j N^{-j} 2^j$$

$$\text{Total Valid Trees} = N^{N-2} \sum_{j=0}^{N/2} \binom{N/2}{j} \left(-\frac{2}{N}\right)^j$$

By the Binomial Theorem, the summation is exactly the expansion of $\left(1 - \frac{2}{N}\right)^{N/2}$. Substituting gives the closed form:

$$\text{Total Valid Trees} = N^{N-2} \left(1 - \frac{2}{N}\right)^{N/2}$$

We can rewrite $(1 - \frac{2}{N})$ as $(\frac{N-2}{N})$. Distributing the exponent yields:

$$\text{Total Valid Trees} = N^{N-2} \frac{(N-2)^{N/2}}{N^{N/2}} = N^{N/2-2} \cdot (N-2)^{N/2}$$

This matches the requested format $N^A \cdot (N-2)^B$, with $A = N/2 - 2$ and $B = N/2$. Hence

$$A + B = (N/2 - 2) + N/2 = N - 2,$$

and substituting $N = 2026$ gives $2026 - 2 = \boxed{2024}$.