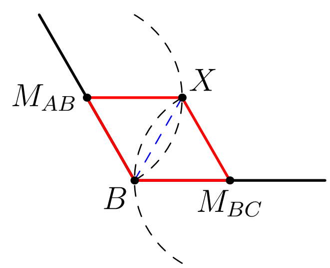
1. *(Brandon Jia)*: On a regular hexagon $ABCDEF$ with side length 2026^{2026}, a circle is drawn on each side such that the side is the diameter of its circle. Repeating this process for all six sides, the circles intersect at six points inside the hexagon. Connecting these points forms a new hexagon $A_1B_1C_1D_1E_1F_1$. Denote this transformation as X.
The process X is applied repeatedly to the resulting hexagons, forming a sequence $A_iB_iC_iD_iE_iF_i$. Determine the largest number of times, a (counting the first transformation shown in the diagram), that X can be applied such that the side length of the resulting hexagon remains an integer. Compute a.

Answer: 2026

Solution: We analyze the transformation X by focusing on a single corner. Let s be the side length of the original hexagon. The circle on side AB and the circle on side BC have radius $s/2$ and are centered at their respective midpoints, M_{AB} and M_{BC}. Their intersection inside the hexagon is point X.



Notice that the quadrilateral $M_{AB}BM_{BC}X$ has all four side lengths equal to $s/2$ (as they are all simply radii of the two identical circles). This makes it a rhombus. A key property of a rhombus is that its diagonals bisect its interior angles. Because the regular hexagon has 120° interior angles, the diagonal BX acts as an angle bisector, meaning X lies on the diagonal connecting B to O.

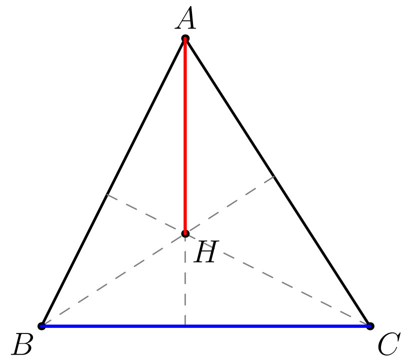
The distance BX is the length of the shorter diagonal of this rhombus. Since the rhombus is comprised of two equilateral triangles glued together, $BX = s/2$. The distance from the center O to B is s. Thus, the distance from the center to our new vertex is $OX = OB - BX = s - s/2 = s/2$. By symmetry, the new hexagon $A_1B_1C_1D_1E_1F_1$ is a regular hexagon with a side length exactly half of the original: $s/2$.

Each application of the transformation X halves the side length. After a applications, the side length becomes $s/2^a$. For this to remain an integer, 2^a must divide 2026^{2026}. Since $2026 = 2 \times 1013$, the largest power of 2 dividing 2026^{2026} is 2^{2026}. Thus, the maximum number of times X can be applied is $\boxed{2026}$.

2. *(Brandon Jia)*: Let $\triangle ABC$ be an acute triangle with orthocenter H. Given that $BC = 2AH$, $AC = 3BH$, and that the area of triangle $\triangle ABC$ is 300, the value of AB can be written as $a\sqrt{b}$, where a and b are positive integers and b is square-free. Compute $a + b$.

Answer: 15

Solution: Let R be the circumradius of $\triangle ABC$. We know that $AH = 2R\cos A$ and by the Extended Law of Sines, $BC = 2R\sin A$.



Given the condition $BC = 2AH$, we can substitute our trigonometric relationships:

$$2R\sin A = 4R\cos A \implies \tan A = 2$$

We can apply the exact same logic to the other given condition. Using $BH = 2R\cos B$ and $AC = 2R\sin B$, the condition $AC = 3BH$ gives us:

$$2R\sin B = 6R\cos B \implies \tan B = 3$$

With two angles known via their tangents, we can then find $\tan C$ using the tangent addition formula.

$$\tan C = -\tan(A+B) = -\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\frac{2+3}{1-6} = 1$$

Since $\triangle ABC$ is an acute triangle, this implies $C = 45°$. We can now express the side lengths a and b in terms of R. Since $\tan A = 2$ and $\tan B = 3$, we draw small right triangles to find $\sin A = \frac{2}{\sqrt{5}}$ and $\sin B = \frac{3}{\sqrt{10}}$.

$$a = 2R\sin A = \frac{4R}{\sqrt{5}}, \quad b = 2R\sin B = \frac{6R}{\sqrt{10}}$$

The area of the triangle is given by the formula $\frac{1}{2}ab\sin C$. Plugging in our expressions:

$$\text{Area} = \frac{1}{2}\left(\frac{4R}{\sqrt{5}}\right)\left(\frac{6R}{\sqrt{10}}\right)\frac{1}{\sqrt{2}} = \frac{24R^2}{2\sqrt{100}} = \frac{12R^2}{10} = \frac{6}{5}R^2$$

We are given that the area of $\triangle ABC$ is 300:

$$\frac{6}{5}R^2 = 300 \implies R^2 = 250 \implies R = 5\sqrt{10}$$

Finally, to compute the side length $c = AB$:

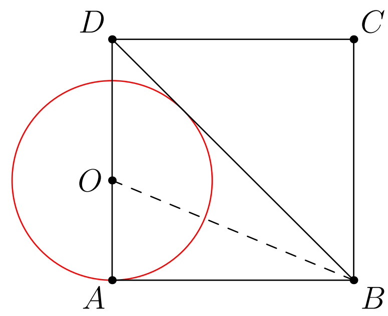
$$c = 2R\sin C = 2(5\sqrt{10})\frac{1}{\sqrt{2}} = 10\sqrt{5}$$

Thus, the length of AB is $10\sqrt{5}$, and the requested answer extraction is $10 + 5 = \boxed{15}$.

3. *(Brandon Jia)*: Square $ABCD$ has side length 10. The center of circle Ω is located within $ABCD$ such that circle Ω is tangent to both line segment AB and diagonal BD. The largest possible radius of the circle can be expressed as $a\sqrt{b}-c$ where a, b, c are positive integers and b is squarefree. What is $a+b+c$?

Answer: 22

Solution:



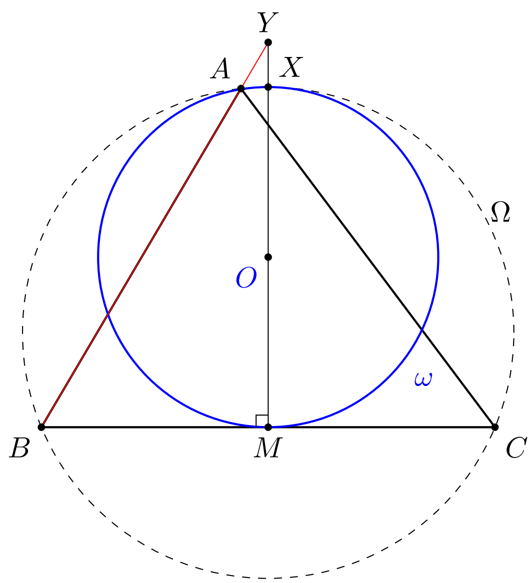
Denote the center of the circle as O. Since O is tangent to both BD and AB, it lies on the angle bisector of $\angle DBA$. As we want to maximize the radius of the circle, the center should be the intersection of the angle bisector and side AD.

Note that $DB = \sqrt{10^2+10^2} = 10\sqrt{2}$. Denote radius of the circle as r. By angle bisector theorem, $\frac{10}{10\sqrt{2}} = \frac{r}{10-r}$. Solving, we get $r = 10(\sqrt{2}-1)$. The requested answer extraction is $\boxed{10+2+10=22}$.

4. *(Brandon Jia)*: Triangle $\triangle ABC$ has side lengths $AB = 13$, $AC = 14$, $BC = 15$. Let M be the midpoint of side BC. Denote the cirumcircle of $\triangle ABC$ as Ω. There exists a circle ω which is internally tangent to Ω at point X and tangent to side BC at M, such that X and A lie on the same side of BC. Denote the center of ω as O. Lines AB and MX intersect at point Y. The value $\sin\angle BYO$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Compute $m+n$.

Answer: 98

Solution:



Because ω is tangent to BC at M, the center of ω, O, must lie on the perpendicular bisector of BC.

The circumcircle Ω of $\triangle ABC$ has its center O' which also lies on the perpendicular bisector of BC. Since the circles ω and Ω are tangent to each other at X, their point of tangency X must be collinear with their centers O and O'.

Because O and O' both lie on the perpendicular bisector of BC, X must also lie on this perpendicular bisector. Thus, the line MX is the perpendicular bisector of BC, and the points Y, X, O, M are collinear.

Because O lies on segment MY, ray YO coincides with ray YM, so $\angle BYO = \angle BYM$. Since $MX \perp BC$, triangle BMY is right-angled at M, giving $\angle BYM = 90° - \angle B$.

Since $\angle BYO = 90° - \angle B$, we have $\sin\angle BYO = \cos B$. It remains to compute $\cos B$ by the Law of Cosines on $\triangle ABC$:

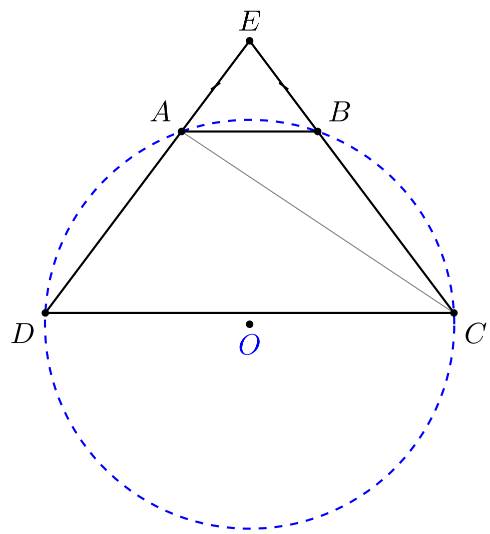
$$\sin\angle BYO = \cos B = \frac{15^2 + 13^2 - 14^2}{2(15)(13)} = \frac{198}{390} = \frac{33}{65}$$

The requested answer extraction is $33 + 65 = \boxed{98}$.

5. *(Brandon Jia)*: Let $ABCD$ be a convex quadrilateral with $AB \parallel CD$. Lines AD and BC are extended past A and B and intersect at point E. Given that $EA = 5$, $ED = 15$, $AB = 6$, and $EA \cdot ED = EB \cdot EC$, the circumradius of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime and b is squarefree. Compute $a + b + c$.

Answer: 20

Solution:



From the given conditions, we note that $ABCD$ is an isosceles trapezoid. Therefore, $EA = EB = 5$. We note that the circumcenter of $\triangle ABC$ is just the circumcenter of the trapezoid. Now:

$$\sin \angle EBA = \frac{4}{5} = \sin \angle ABC$$

We can find $\cos \angle E$:

$$\cos(\angle E) = 2\cos^2\left(\frac{\angle E}{2}\right) - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{7}{25}$$

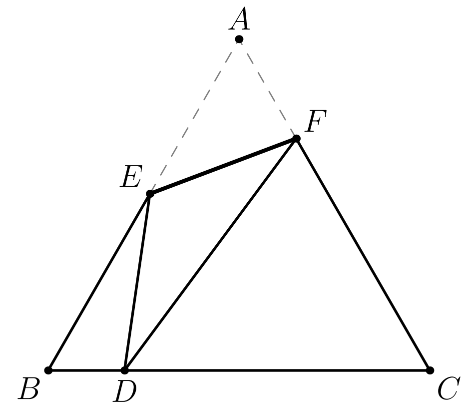
Using Law of Cosines on $\triangle EAC$, we find that $AC = 4\sqrt{13}$. We can now apply Extended Law of Sines:

$$2R = \frac{AC}{\sin(\angle ABC)}$$

$$2R = 5\sqrt{13} \implies R = \frac{5\sqrt{13}}{2}$$

The requested answer extraction is $5 + 13 + 2 = \boxed{20}$.

6. *(Brandon Jia)*: An equilateral triangle piece of paper $\triangle ABC$ with side length 15 is folded such that vertex A lands exactly on a point D on side BC. The crease line intersects AB at E and AC at F. Given that $BD = 3$, the area of $\triangle AEF$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is square-free. Compute $a + b + c$.



Answer: 158

Solution: When the paper is folded, the triangle $\triangle AEF$ maps onto $\triangle DEF$. The crease line acts as a line of reflection. Therefore, any segment connected to A maps to a segment of the exact same length connected to D.

Let $x = BE$. The remaining part of that side is $AE = 15 - x$. Because AE folds onto DE, we have $DE = 15 - x$. For $\triangle BDE$, we know the angle $\angle B = 60^\circ$ and $BD = 3$. Using Law of Cosines:

$$DE^2 = BE^2 + BD^2 - 2(BE)(BD)\cos 60^\circ \implies (15 - x)^2 = x^2 + 9 - 3x$$

Expanding and solving for x:

$$225 - 30x + x^2 = x^2 - 3x + 9 \implies 27x = 216 \implies x = 8$$

So $BE = 8$, which means $AE = 7$. We repeat the same process on the right side. Let $y = CF$. The segment AF maps to DF, so $DF = AF = 15 - y$. The bottom segment is $CD = BC - BD = 15 - 3 = 12$. Using the Law of Cosines on $\triangle CDF$:

$$DF^2 = CF^2 + CD^2 - 2(CF)(CD)\cos 60^\circ \implies (15 - y)^2 = y^2 + 144 - 12y$$

$$225 - 30y + y^2 = y^2 - 12y + 144 \implies 18y = 81 \implies y = \frac{9}{2}$$

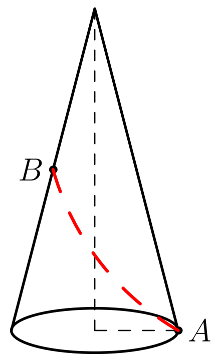
Therefore, $CF = 4.5$, leaving $AF = 10.5 = \frac{21}{2}$.

Now that we have both sides of $\triangle AEF$ meeting at the 60° angle at A, we can find the area:

$$\text{Area} = \frac{1}{2}(AE)(AF)\sin 60^\circ = \frac{1}{2}(7)\left(\frac{21}{2}\right)\frac{\sqrt{3}}{2} = \frac{147\sqrt{3}}{8}$$

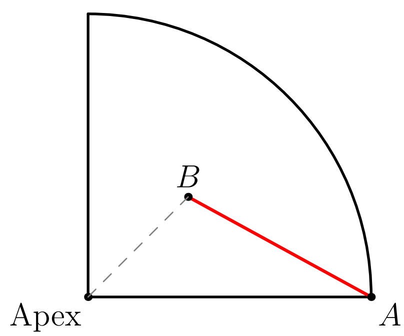
This matches the format $\frac{a\sqrt{b}}{c}$ with $a = 147$, $b = 3$, and $c = 8$. The requested sum is $147 + 3 + 8 = \boxed{158}$.

7. *(Brandon Jia)*: A solid right circular cone has a base with radius 3 and a slant height of 12. An ant starts at point A on the circumference of the base. The ant wishes to crawl along the lateral surface of the cone to reach a point B, which is located diametrically opposite of A relative to the base, and sits exactly halfway up the slant height of the cone. The square of the shortest distance the ant can travel can be expressed as $m - n\sqrt{p}$, where m, n, and p are integers and p is square-free. Find $m + n + p$.



Answer: 254

Solution: The idea is to unroll the lateral surface of the cone into the plane. This flattens the cone into a circular sector whose radius equals the cone's slant height, 12. The arc length of the unrolled sector is equal to the circumference of the cone's base, which is $2\pi r = 6\pi$.



We find the central angle of this sector:

$$\theta = \frac{\text{arc length}}{L} = \frac{6\pi}{12} = \frac{\pi}{2} = 90^\circ$$

Let the apex of the cone be the origin of a polar coordinate system. The ant starts at point A on the boundary of the sector, which we can orient at the polar coordinates $(12, 0^\circ)$.

Point B is diametrically opposite A on the cone's base. This means walking to the opposite side covers exactly half the circumference, or 3π. In our unrolled 2D sector, this corresponds to an angle of $\frac{3\pi}{12} = \frac{\pi}{4} = 45^\circ$. Since B is halfway up the slant height, its distance from the apex is 6. Thus, point B maps to the polar coordinates $(6, 45^\circ)$.

The shortest distance d between A and B on the 3D cone is a straight line in the 2D plane. We can thus find the distance using the Law of Cosines on the polar triangle formed by the apex, A, and B:

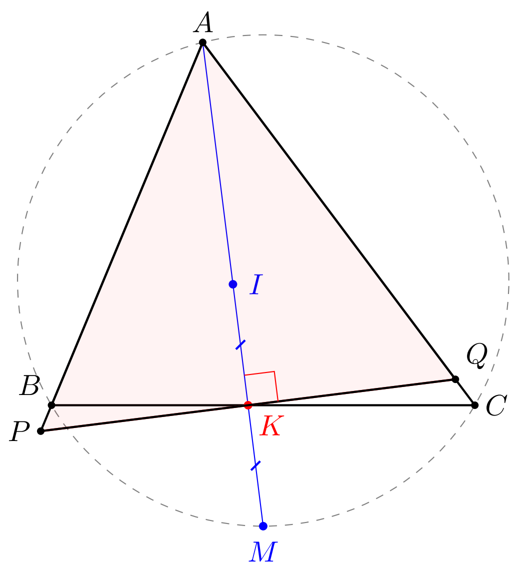
$$d^2 = 12^2 + 6^2 - 2(12)(6)\cos 45^\circ = 144 + 36 - 144\left(\frac{\sqrt{2}}{2}\right) = 180 - 72\sqrt{2}$$

This is in the exact requested form $m - n\sqrt{p}$, with $m = 180$, $n = 72$, and $p = 2$. Their sum is $180 + 72 + 2 = \boxed{254}$.

8. *(Brandon Jia)*: In triangle $\triangle ABC$, let the side lengths be $AB = 13$, $BC = 14$, and $AC = 15$. Let I be the incenter of the triangle, and let the ray AI intersect the circumcircle of $\triangle ABC$ at a point M (where $M \neq A$). Let K be the midpoint of the segment IM. A line ℓ is drawn through K such that ℓ is perpendicular to the line AM. This line ℓ intersects the lines AB and AC at points P and Q, respectively. The area of $\triangle APQ$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Compute $m + n$.

Answer: 592

Solution:



By the incenter–excenter lemma, M is the center of a circle passing through B, I, C, and the A-excenter. We can compute:

$$IM = 2R\sin\left(\frac{A}{2}\right)$$

Similarly, the distance from a vertex to the incenter is:

$$AI = \frac{r}{\sin\left(\frac{A}{2}\right)}$$

From Law of Cosines on $\triangle ABC$, we find that $\cos\angle A = \frac{33}{65}$. Using the half angle identity for sin, we can find:

$$\sin^2\left(\frac{A}{2}\right) = \frac{1 - \frac{33}{65}}{2} = \frac{16}{65} \implies \sin\left(\frac{A}{2}\right) = \frac{4}{\sqrt{65}}$$

We now calculate IM and AI, and find that:

$$IM = AI = \sqrt{65}$$

Since $\ell \perp AM$ at K, and AM is the angle bisector of $\angle PAQ$, triangle APQ is isosceles with $AP = AQ$. The segment AK is the altitude to the base PQ.
Since K is the midpoint of IM, and $AI = IM = \sqrt{65}$:

$$AK = AI + IK = \sqrt{65} + \frac{1}{2}\sqrt{65} = \frac{3}{2}\sqrt{65}$$

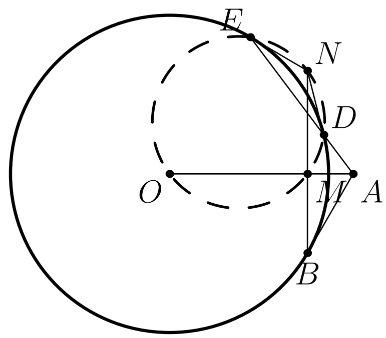
In the right triangle APK, the base $PK = AK\tan\left(\frac{A}{2}\right)$.
Using $\sin\left(\frac{A}{2}\right) = \frac{4}{\sqrt{65}}$, we find $\cos\left(\frac{A}{2}\right) = \frac{7}{\sqrt{65}}$, so $\tan\left(\frac{A}{2}\right) = \frac{4}{7}$.

$$|APQ| = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2}(2 \cdot PK)(AK) = AK^2 \tan\left(\frac{A}{2}\right)$$

$$|APQ| = \left(\frac{3}{2}\sqrt{65}\right)^2 \cdot \frac{4}{7} = \frac{9}{4} \cdot 65 \cdot \frac{4}{7} = \frac{585}{7}$$

The requested answer extraction is $585 + 7 = \boxed{592}$.

9. *(Brandon Jia)*: Given a circle Γ with center O and radius 780, let A be a point such that $OA = 900$. The two tangents from A to Γ touch the circle at points B and C. A line through A intersects Γ at points D and E such that D lies between A and E, and $AD = 240$. Let M be the midpoint of BC. The circumcircle of $\triangle MDE$ intersects the line BC at M and another point N. Compute the length of MN.



Answer: 507

Solution: Let $R = 780$ be the radius of the circle Γ. The line segment BC is the chord of contact from A to Γ, which means $BC \perp OA$ at its midpoint M. Because the radius OB is perpendicular to the tangent AB, $\triangle OBA$ is a right triangle. The segment BM is the altitude to the hypotenuse OA, so by the geometric mean theorem (or by the similar triangles $\triangle OBM \sim \triangle OBA$), we establish that $OB^2 = OM \cdot OA$. Substituting the known values gives $780^2 = OM \cdot 900$, which yields:

$$OM = 676$$

By definition, the power of a point outside a circle is the square of the distance to the center minus the square of the radius, so $P(A, \Gamma) = OA^2 - R^2 = 900^2 - 780^2 = 201600$. Because A, D, and E form a secant line to the circle, the product of the secant segments equals the power of the point, so $AD \cdot AE = P(A, \Gamma)$. Given $AD = 240$:

$$240 \cdot AE = 201600$$

$$AE = 840$$

We can also express the power of point A using the segment OA. By factoring, $OA^2 - R^2 = OA^2 - (OM \cdot OA) = OA(OA - OM) = OA \cdot AM$. This establishes that $AD \cdot AE = AM \cdot AO$, which means that $\triangle ADM$ and $\triangle AOE$ are similar, which means that the quadrilateral $OMDE$ is cyclic.

Therefore, points O, M, D, and E all lie on the same circle, meaning the circumcircle of $\triangle MDE$ must also pass through the center O.

We are given that this circumcircle intersects the line BC at N. Because $BC \perp OA$ at M, the inscribed angle $\angle OMN$ is $90°$. According to Thales's theorem, any $90°$ inscribed angle must subtend a diameter, meaning the line segment ON is a diameter of this circumcircle.

The angles subtended by the diameter ON at points D and E must also be right angles, so $\angle ODN = \angle OEN = 90°$. Because OD and OE are radii of Γ, the lines ND and NE being perpendicular to the radii means they must be tangent to Γ at D and E. This means that N is the pole of the secant line ADE with respect to Γ.

Because tangents from an external point are symmetric, the line connecting the center to the external point (ON) is the perpendicular bisector of the chord of contact (DE). Let P be the intersection of ON and DE, making P the midpoint of DE. The length of the chord DE

is $AE - AD = 840 - 240 = 600$, so $DP = 300$. Using the Pythagorean theorem in the right triangle $\triangle OPD$, we find:

$$OP = \sqrt{OD^2 - DP^2} = \sqrt{780^2 - 300^2} = 720$$

In the right triangle $\triangle ODN$, DP is the altitude to the hypotenuse ON. Using the geometric mean theorem again, we have $OD^2 = OP \cdot ON$. Substituting our values gives:

$$780^2 = 720 \cdot ON$$

Which simplifies to:

$$ON = 845$$

Lastly, we apply the Pythagorean theorem in the right triangle $\triangle OMN$ to find the length of MN. We have $MN = \sqrt{ON^2 - OM^2} = \sqrt{845^2 - 676^2}$. Factoring this as a difference of squares gives:

$$\sqrt{(845 - 676)(845 + 676)} = \sqrt{169 \cdot 1521} = 13 \cdot 39 = \boxed{507}$$

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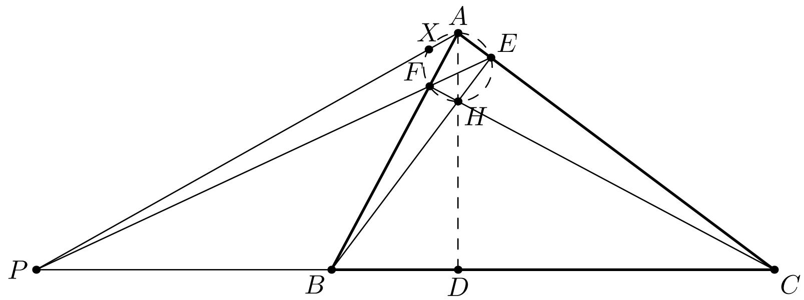
10. *(Brandon Jia)*: Let $\triangle ABC$ be an acute triangle with side lengths $AB = 17$, $AC = 25$, and $BC = 28$. Let H be the orthocenter of $\triangle ABC$. Let E and F be the feet of the altitudes from B and C to the sides AC and AB, respectively. The line EF intersects the line BC at point P. The line AP intersects the circumcircle of $\triangle AEF$ at a second point X (where $X \neq A$). The value of AX^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.



Answer: 1858

Solution: We consider the powers of point P with respect to the two relevant circles in this configuration. Let ω_1 be the circumcircle of $\triangle AEF$ (which has diameter AH) and ω_2 be the circle with diameter BC containing the concyclic points B, C, E, F. EF acts as the radical axis of these two circles. Since P lies on EF, the power of P with respect to both circles must be equal, which gives $PE \cdot PF = PB \cdot PC$.

Because X is on ω_1, the angle inscribed in the semicircle $\angle AXH = 90°$, making X the projection of H onto AP. Additionally, the power of P with respect to ω_1 is $PA \cdot PX$. Thus, chaining our power equations together, we find $PA \cdot PX = PB \cdot PC$. This means that A, B, X, C are concyclic, which means that X is on the circumcircle of $\triangle ABC$.

By Menelaus's Theorem on $\triangle ABC$ with the transversal line PEF:

$$\frac{PB}{PC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Using side lengths $17, 25, 28$, we can find the cosines of the angles: $\cos C = \frac{4}{5}$ and $\cos B = \frac{8}{17}$. This lets us find the segments on the sides: $CE = a \cos C = \frac{112}{5}$ and $EA = 25 - \frac{112}{5} = \frac{13}{5}$. Similarly, $BF = a \cos B = \frac{224}{17}$ and $FA = 17 - \frac{224}{17} = \frac{65}{17}$. Plugging this back in to our Menelaus relation:

$$\frac{PB}{PC} \cdot \frac{112/5}{13/5} \cdot \frac{65/17}{224/17} = 1 \implies \frac{PB}{PC} = \frac{2}{5}$$

Since P is exterior to BC, $PC = PB + 28$. We deduce $PB = \frac{56}{3}$ and $PC = \frac{140}{3}$. The power of P is $PB \cdot PC = \frac{7840}{9}$. Let D be the foot of the altitude from A. $BD = 8$ and $AD = 15$. The total distance $PD = PB + BD = \frac{80}{3}$. Using the Pythagorean theorem on the large right triangle $\triangle PAD$:

$$PA^2 = PD^2 + AD^2 = \left(\frac{80}{3}\right)^2 + 15^2 = \frac{8425}{9}$$

Since X lies on segment PA, we have $AX = PA - PX$. Multiplying by PA:

$$PA \cdot AX = PA^2 - PA \cdot PX = \frac{8425}{9} - \frac{7840}{9} = \frac{585}{9} = 65$$

Thus, $AX = \frac{65}{PA}$, giving $AX^2 = \frac{4225}{8425/9} = \frac{1521}{337}$. Since 1521 and 337 are relatively prime (we can determine this by the Euclidian Algorithm), $m = 1521$ and $n = 337$, leaving $m + n = \boxed{1858}$.