1. *(Brandon Jia)*: A certain four-digit house number N is a perfect square. It has the property that if each of its digits is increased by 1, the resulting four-digit number is also a perfect square. Determine the value of N.

   Answer: 2025

   Solution: Let the original four-digit number be $N = x^2$. Increasing each of its digits by 1 is equivalent to adding 1111 to the number. We are given that this new number is also a perfect square, which we can call y^2. We can set up the following equation:

$$y^2 = x^2 + 1111$$

$$y^2 - x^2 = 1111$$

$$(y - x)(y + x) = 1111$$

   The prime factorization of 1111 is 11×101, and 101 is prime.

   Because x and y are positive integers, we must have $y+x > y-x$. Therefore, the only possible assignment for the factors is:

$$y - x = 11$$

$$y + x = 101$$

   Adding the two equations together yields $2y = 112$, meaning $y = 56$. Subtracting them yields $2x = 90$, meaning $x = 45$. We want the original house number N, which is $x^2 = 45^2 = \boxed{2025}$.

2. *(Christopher Liang)*: How many integers $x < 2026$ are divisible by exactly two of either 2, 3, 5, or 7?

Answer: 539

Solution: We are looking for the number of positive integers up to 2025 that are multiples of exactly two primes from the set $\{2, 3, 5, 7\}$.

Let $N(d)$ denote the number of positive integers up to 2025 that are divisible by d. This is simply $N(d) = \lfloor 2025/d \rfloor$. By the Principle of Inclusion-Exclusion (PIE), the number of integers satisfying exactly two properties out of a given set of conditions is found using the formula:

$$\text{Exactly } 2 = \sum N(p_i p_j) - 3 \sum N(p_i p_j p_k) + 6N(p_1 p_2 p_3 p_4)$$

First, we compute the counts for the pairs:

- $N(6) = \lfloor 2025/6 \rfloor = 337$
- $N(10) = \lfloor 2025/10 \rfloor = 202$
- $N(14) = \lfloor 2025/14 \rfloor = 144$
- $N(15) = \lfloor 2025/15 \rfloor = 135$
- $N(21) = \lfloor 2025/21 \rfloor = 96$
- $N(35) = \lfloor 2025/35 \rfloor = 57$

The sum of the pair counts is $337 + 202 + 144 + 135 + 96 + 57 = 971$.

Next, we compute the counts for the triplets:

- $N(30) = \lfloor 2025/30 \rfloor = 67$
- $N(42) = \lfloor 2025/42 \rfloor = 48$
- $N(70) = \lfloor 2025/70 \rfloor = 28$
- $N(105) = \lfloor 2025/105 \rfloor = 19$

The sum of the triplet counts is $67 + 48 + 28 + 19 = 162$.

Finally, the quadruplet count is just $N(210) = \lfloor 2025/210 \rfloor = 9$.

Plugging these into our PIE formula:

$$\text{Exactly } 2 = 971 - 3(162) + 6(9) = 971 - 486 + 54 = \boxed{539}$$

3. *(Brandon Jia)*: Let $n = 2^{12} \cdot 5^{12}$. A positive divisor d of n is called *equidivisive* if $\tau(d) = \tau(n/d)$, where $\tau(k)$ denotes the number of positive divisors of k. Determine the number of equidivisive divisors of n.

Answer: 13

Solution: Any divisor d of $n = 2^{12} \cdot 5^{12}$ can be written in the form $d = 2^a 5^b$, where $0 \le a \le 12$ and $0 \le b \le 12$. The co-divisor is then $n/d = 2^{12-a} 5^{12-b}$.

We can find the number of divisors of d and n/d:

$$\tau(d) = (a+1)(b+1)$$

$$\tau(n/d) = (13-a)(13-b)$$

We are given that $\tau(d) = \tau(n/d)$, so we can equate them and expand:

$$ab + a + b + 1 = 169 - 13a - 13b + ab$$

We can simplify:

$$14a + 14b = 168$$

$$a + b = 12$$

We are looking for pairs of integers (a, b) such that $0 \le a \le 12$, $0 \le b \le 12$, and $a + b = 12$. For every choice of $a \in \{0, 1, 2, \ldots, 12\}$, there is exactly one corresponding valid b (which is $12 - a$).

Counting the possible values for a, we find there are exactly $\boxed{13}$ such equidivisive divisors.

4. *(Brandon Jia)*: Determine the number of positive integers $n \leq 2026$ such that $\lfloor \sqrt[3]{n} \rfloor$ is a divisor of n.

Answer: 267

Solution: Let $k = \lfloor \sqrt[3]{n} \rfloor$. By the definition of the floor function, this means $k^3 \leq n < (k+1)^3$. We are given the condition that k divides n, so we can write $n = k \cdot m$ for some positive integer m. Substituting this back into our inequality bounds:

$$k^3 \leq km \leq (k+1)^3 - 1$$

$$k^3 \leq km \leq k^3 + 3k^2 + 3k$$

Dividing the entire inequality by k, we find the bounds for m:

$$k^2 \leq m \leq k^2 + 3k + 3$$

For a fixed integer k, the number of valid multiples m is simply the upper bound minus the lower bound plus 1:

$$(k^2 + 3k + 3) - k^2 + 1 = 3k + 4$$

Now we need to consider how high k can go. We are given $n \leq 2026$. Notice that $12^3 = 1728$ and $13^3 = 2197$. This means k goes all the way up to 12, but for $k = 12$, the values of n will get cut off before reaching $13^3 - 1$.

For $k = 1, 2, \ldots, 11$, all $3k + 4$ values fall within our range. Summing:

$$\sum_{k=1}^{11} (3k + 4) = 3\frac{11 \cdot 12}{2} + 11 \cdot 4 = 198 + 44 = 242$$

For $k = 12$, we need $n = 12m \leq 2026$. This sets a strict upper bound on m:

$$m \leq \lfloor 2026/12 \rfloor = 168$$

The lower bound for m when $k = 12$ is $12^2 = 144$. So m can be any integer from 144 to 168 inclusive. The number of such values is $168 - 144 + 1 = 25$.

The total number of valid integers n is $242 + 25 = \boxed{267}$.

5. *(Brandon Jia)*: For a positive integer n, let $s_b(n)$ denote the sum of the digits of n when written in base b. Determine the number of integers n in the set $\{1, 2, 3, \dots, 1000\}$ that satisfy the equation

$$s_2(n) = s_4(2n)$$

Answer: 31

Solution: We first examine how bases 2 and 4 represent numbers. A base-4 digit corresponds to a pair of binary bits. Let the binary representation of $2n$ be $\dots x_3x_2x_1x_0$. Because $2n$ is even, its last bit $x_0 = 0$. The bits of n are exactly the bits of $2n$ shifted down by one place: $x_i = y_{i-1}$, so the sum of the digits of n in base 2 is the same as the sum of the digits of $2n$ in base 2:

$$s_2(n) = \sum_{j \geq 0} (x_{2j+1} + x_{2j})$$

When we write $2n$ in base 4, each digit is formed by a pair of binary bits. The j-th base 4 digit has the value $2x_{2j+1} + x_{2j}$. Thus:

$$s_4(2n) = \sum_{j \geq 0} (2x_{2j+1} + x_{2j})$$

We are given that $s_2(n) = s_4(2n)$, so we can equate the sums:

$$\sum_{j \geq 0} (x_{2j+1} + x_{2j}) = \sum_{j \geq 0} (2x_{2j+1} + x_{2j})$$

Subtracting the left side from the right leaves us with:

$$\sum_{j \geq 0} x_{2j+1} = 0$$

Since all bits x_i must be 0 or 1, this implies that $x_{2j+1} = 0$ for all $j \geq 0$. This means that in the binary representation of $2n$, every odd-indexed bit must be 0.

Translating this back to n, since n is just $2n$ shifted by one position, every *even-indexed* bit of n must be 0. Therefore, n can only have 1's in its odd-indexed binary positions (i.e., the $2^1, 2^3, 2^5, 2^7 \dots$ places). This means n must be formed by a sum of a subset of the numbers $\{2, 8, 32, 128, 512, 2048, \dots\}$.

We are restricted to $n \leq 1000$. The maximum subset we can form using the valid powers up to 1000 is $\{2, 8, 32, 128, 512\}$. The sum of all five of these numbers is 682, which safely remains below 1000.

There are 5 available powers of 2. We can either include or exclude each one, giving $2^5 = 32$ possible combinations. However, the problem asks for integers $n \in \{1, 2, \dots, 1000\}$, and picking the empty set would give $n = 0$, which is excluded. This leaves us with $32 - 1 = \boxed{31}$ numbers.

6. *(Christopher Liang)*: The function $f(x, y)$ is defined for all integers $1 < x \leq y$. For all primes p and integers a, b, $c > 1$ the function f has the following properties:

$$f(a^b, c) = f(a, c)$$

$$f(p, p) = 1, f(a, b) = 0 \text{ if gcd(a,b)=1}$$

$$f(p, bc) = f(p, b) + f(p, c)$$

$$f(ab, c) = \max(f(a, c), f(b, c)) \text{ if gcd(a,b)=1}$$

Determine the sum of all integers $x > 1$ such that

$$f(x, 2026) \geq f(k, 2026)$$

for every integer $1 < k \leq 2026$.

Answer: 1028195

Solution: We first determine what the function f represents. Applying the third rule, $f(p, bc) = f(p, b) + f(p, c)$, tells us that for a fixed prime p, the function behaves logarithmically with respect to its second argument. Specifically, $f(p, y)$ counts the number of times p divides y. That is, $f(p, y) = v_p(y)$, the p-adic valuation of y.

Now we evaluate x. If $x = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, we can use the fourth rule to split coprime factors:

$$f(x, y) = \max_{p_i | x} (f(p_i^{e_i}, y))$$

Using the first rule, $f(p_i^{e_i}, y) = f(p_i, y) = v_{p_i}(y)$. So the function evaluates to the maximum exponent among the prime factors of x in the prime factorization of y:

$$f(x, y) = \max_{p | x} v_p(y)$$

We need to maximize $f(x, 2026)$ over all $x \leq 2026$.

Note that the prime factorization of 2026 is 2×1013. Therefore, $v_2(2026) = 1$ and $v_{1013}(2026) = 1$. For any other prime q, $v_q(2026) = 0$.

For any $k \leq 2026$, $f(k, 2026)$ can be at most 1. We want x to achieve this maximum value, meaning we must have $f(x, 2026) = 1$. This happens if and only if x has either 2 or 1013 as a prime factor (i.e., x is a multiple of 2 or a multiple of 1013).

We need to find the sum of all such $x \in (1, 2026]$. To do this, we can use PIE:

- Sum of multiples of 2: The sum of even numbers up to 2026 is $2(1 + 2 + \cdots + 1013) = 1013 \times 1014 = 1,027,182$.
- Sum of multiples of 1013: The only multiples in our range are 1013 and 2026, which sum to 3039.
- Sum of multiples of both (2026): The only multiple is 2026.

Thus, the total sum is $= 1,027,182 + 3039 - 2026 = \boxed{1028195}$.

7. *(Brandon Jia)*: For a positive integer n, let $\phi(n)$ denote the number of integers $1 \le k \le n$ such that $\gcd(k, n) = 1$. Determine the number of integers $n \in \{1, 2, \dots, 100\}$ such that $\gcd(n, \phi(n)) > 1$.

Answer: 63

Solution: We are looking for numbers where n and its totient $\phi(n)$ share a prime factor. It is easier to count the complement: the integers $n \le 100$ where $\gcd(n, \phi(n)) = 1$. Numbers satisfying this condition are known as *cyclic numbers.*

A number n is cyclic if and only if it is square-free ($n = p_1 p_2 \dots p_k$) and no prime factor p_i divides $(p_j - 1)$ for any other prime factor p_j.

- $n = 1$: $\phi(1) = 1$, and $\gcd(1, 1) = 1$. (1 number)
- For any prime p, $\phi(p) = p - 1$. Since p is prime, it cannot divide $p - 1$. There are exactly 25 primes under 100. (25 numbers)
- For products of two primes (pq, with $p < q$): We need $p \nmid (q - 1)$.
  - If $p = 2$, then $q - 1$ is even, so 2 will always divide $q - 1$. No cyclic numbers here.
  - If $p = 3$, we need $q \not\equiv 1 \pmod 3$, so $q \equiv 2 \pmod 3$. Valid products ≤ 100 are $3 \times 5 = 15$, $3 \times 11 = 33$, $3 \times 17 = 51$, $3 \times 23 = 69$, $3 \times 29 = 87$. (5 numbers)
  - If $p = 5$, we need $q \not\equiv 1 \pmod 5$. Valid products are $5 \times 7 = 35$, $5 \times 13 = 65$, $5 \times 17 = 85$, $5 \times 19 = 95$. (4 numbers)
  - If $p = 7$, we need $q \not\equiv 1 \pmod 7$. Valid products are $7 \times 11 = 77$, $7 \times 13 = 91$. (2 numbers)
- For products of three primes: The smallest odd square-free product of three primes is $3 \times 5 \times 7 = 105 > 100$. (0 numbers)

The total count of cyclic numbers up to 100 is $1 + 25 + 5 + 4 + 2 = 37$. Since we are looking for the numbers that are *not* cyclic, we subtract this from the total: $100 - 37 = \boxed{63}$.

8. *(Christopher Liang)*: Positive integers a, b, c satisfy the following equation:

$$a(a^2 + 3b^2) = 4(5c - a)^3$$

How many ordered triplets (a, b, c) are there for $a \leq 2026$?

Answer: 405

Solution: We simplify the left-hand side. Observe that:

$$(a+b)^3+(a-b)^3 = (a^3+3a^2b+3ab^2+b^3)+(a^3-3a^2b+3ab^2-b^3) = 2a^3+6ab^2 = 2a(a^2+3b^2)$$

So $a(a^2 + 3b^2) = \frac{(a+b)^3+(a-b)^3}{2}$. Substituting back into the original equation:

$$\frac{(a+b)^3 + (a-b)^3}{2} = 4(5c - a)^3$$

$$(a + b)^3 + (a - b)^3 = 8(5c - a)^3 = (10c - 2a)^3$$

This is Fermat's Last Theorem for $n = 3$, which states that the equation $X^3 + Y^3 = Z^3$ has no non-trivial integer solutions. Thus, one of the terms must be zero.

Since a and b are strictly positive integers, $a + b > 0$, so $X \neq 0$. If $10c - 2a = 0$, then $X^3 + Y^3 = 0 \implies X = -Y \implies a + b = -(a - b) \implies 2a = 0$, which is impossible. Therefore, the only way this works is if the middle term is zero: $a - b = 0$, which means $a = b$.

If $a = b$, the equation simplifies to:

$$(2a)^3 + 0 = (10c - 2a)^3$$

$$2a = 10c - 2a$$

$$4a = 10c \implies 2a = 5c$$

For a and c to be positive integers, a must be a multiple of 5. We can parameterize the solutions as $a = 5k$, which dictates that $c = 2k$ and $b = 5k$ for any positive integer k.

We are given the bound $a \leq 2026$.

$$5k \leq 2026 \implies k \leq \lfloor 2026/5 \rfloor = 405$$

Thus, there are exactly $\boxed{405}$ valid ordered triplets (a, b, c).

9. *(Brandon Jia)*: A token is placed at the position 0 on a number line. For a fixed positive integer n, the token performs a sequence of n jumps. In the k-th jump, for $k = 1, 2, \dots, n$, the token moves from its current position x to a new position $x + \lceil n/k \rceil$. Let L_n denote the final position of the token after all n jumps have been completed. Determine the number of positive integers $n \leq 500$ such that the final position L_n satisfies the equation:

$$L_n = \sum_{k=1}^{n} \lfloor n/k \rfloor + (n - 8)$$

Answer: 87

Solution: The token's final position is the sum of all its jumps: $L_n = \sum_{k=1}^{n} \lceil n/k \rceil$. Substituting this into our given condition:

$$\sum_{k=1}^{n} \lceil n/k \rceil - \sum_{k=1}^{n} \lfloor n/k \rfloor = n - 8$$

$$\sum_{k=1}^{n} (\lceil n/k \rceil - \lfloor n/k \rfloor) = n - 8$$

Consider the quantity $\lceil n/k \rceil - \lfloor n/k \rfloor$. If k divides n, then n/k is an integer, making the ceiling and floor identical, so their difference is 0. If k does not divide n, n/k is a fraction, so the ceiling is exactly 1 higher than the floor.

Therefore, the summation on the left simply counts the number of integers $k \in \{1, 2, \dots, n\}$ that do *not* divide n. If we let $\tau(n)$ be the number of divisors of n, the number of non-divisors is $n - \tau(n)$. Our equation becomes:

$$n - \tau(n) = n - 8 \implies \tau(n) = 8$$

We need to find the number of integers up to 500 that have exactly 8 divisors. The prime factorization of such numbers must be in one of three forms (based on the ways to write 8 as a product of integers > 1: 8, 4×2, or $2 \times 2 \times 2$):

- p^7: $2^7 = 128 \leq 500$. (Next is $3^7 = 2187$). This gives 1 number.
- $p^3 q$:
  - $p = 2$: $8q \leq 500 \implies q \leq 62$. There are 17 primes up to 62 excluding 2.
  - $p = 3$: $27q \leq 500 \implies q \leq 18$. There are 6 primes up to 18 excluding 3.
  - $p = 5$: $125q \leq 500 \implies q \leq 4$. There are 2 primes up to 4 excluding 5 (i.e. 2, 3).
  - $p = 7$: $343q \leq 500 \implies q \leq 1$. No primes.

  This gives $17 + 6 + 2 = 25$ numbers.
- pqr $(p < q < r)$:
  - $p = 2$: $qr \leq 250$. For $q = 3$ ($r \leq 83$, 21 primes). For $q = 5$ ($r \leq 50$, 12 primes). For $q = 7$ ($r \leq 35$, 7 primes). For $q = 11$ ($r \leq 22$, 3 primes). For $q = 13$ ($r \leq 19$, 2 primes). Total = 45.
  - $p = 3$: $qr \leq 166$. For $q = 5$ ($r \leq 33$, 8 primes). For $q = 7$ ($r \leq 23$, 5 primes). For $q = 11$ ($r \leq 15$, 1 prime). Total = 14.
  - $p = 5$: $qr \leq 100$. For $q = 7$ ($r \leq 14$, 2 primes). Total = 2.

This gives $45 + 14 + 2 = 61$ numbers.

Summing them up, we get a total of $1 + 25 + 61 = \boxed{87}$ numbers.

10. *(Brandon Jia)*: For each positive integer n, let $f(n) = \sum_{k=1}^{n} \gcd(k, n)$. Determine the sum of all positive integers n such that $f(n) = 3n$.

Answer: 58

Solution: It is a standard number theory identity that $f(n) = \sum_{k=1}^{n} \gcd(k, n) = \sum_{d|n} d\phi(n/d)$, where the summation runs over all divisors d of n. We can re-index the summation by replacing d with n/d, giving us $f(n) = \sum_{d|n} \frac{n}{d}\phi(d) = n\sum_{d|n} \frac{\phi(d)}{d}$.

We are given $f(n) = 3n$. Dividing by n, we obtain:

$$\sum_{d|n} \frac{\phi(d)}{d} = 3$$

Let $g(n) = \sum_{d|n} \frac{\phi(d)}{d}$. Since $\phi(d)$ and $1/d$ are both multiplicative functions, $g(n)$ is also multiplicative. We can evaluate it on prime powers p^k:

$$g(p^k) = \sum_{j=0}^{k} \frac{\phi(p^j)}{p^j} = 1 + \sum_{j=1}^{k} \frac{p^{j-1}(p-1)}{p^j} = 1 + k\left(1 - \frac{1}{p}\right) = \frac{p + k(p-1)}{p}$$

Because g is multiplicative, for $n = \prod p_i^{k_i}$, we need $\prod \frac{p_i + k_i(p_i - 1)}{p_i} = 3$.

We first test single prime powers: $g(p^k) = 3 \implies \frac{p+k(p-1)}{p} = 3 \implies k(p-1) = 2p$. Since $\gcd(p-1, p) = 1$, $p - 1$ must divide 2. This restricts us to $p - 1 = 1$ (so $p = 2$) or $p - 1 = 2$ (so $p = 3$).

- If $p = 2$, $k(1) = 4 \implies k = 4$. This yields $n = 2^4 = 16$.
- If $p = 3$, $k(2) = 6 \implies k = 3$. This yields $n = 3^3 = 27$.

Now, what if n has multiple distinct prime factors? Let $n = 2^a \cdot m$, where m is odd. Then $g(n) = g(2^a)g(m) = 3$.

$$g(2^a) = \frac{2+a}{2}, \quad g(m) = \frac{A}{B}$$

where $B = \prod_{p|m} p$ is an odd number, and $A = \prod_{p|m}(p + k(p-1))$. Because p is odd and $p-1$ is even, $p + k(p-1)$ is strictly odd, meaning A is an odd number. Equating the product to 3:

$$\frac{2+a}{2} \cdot \frac{A}{B} = 3 \implies (2+a)A = 6B$$

Since A and B are both odd, the right side $6B$ has exactly one factor of 2. Therefore, the left side must also have exactly one factor of 2, meaning $2 + a$ must be equivalent to 2 (mod 4). This requires a to be a multiple of 4. Let $a = 4c$.

- If $c = 1$, $a = 4$, then $6A = 6B \implies A = B$. This implies $m = 1$, giving us $n = 16$, which we already found.
- If $c > 1$, $a \geq 8$, then $g(2^a) \geq 5$. Since $g(m) \geq 1$, their product would be greater than 3.
- If $c = 0$, $a = 0$, meaning n has no factors of 2. Then $A = 3B$.

If $A = 3B$, n is purely comprised of odd primes, and $g(m) = 3$. We found $m = 27$ earlier. Now, we finally need to ask whether there are combinations of multiple primes.

Notice that evaluating small primes gives: $g(3) = 5/3$, $g(5) = 9/5$.

Multiplying these together yields $g(15) = g(3)g(5) = (5/3)(9/5) = 3$. This gives $n = 15$.

Any other primes $p \geq 5$ produce a factor strictly $\geq 9/5 = 1.8$. If $p = 3$ was not used, we'd need multiple large primes to hit 3, but $1.8 \times 1.8 = 3.24 > 3$. So $p = 3$ must be included.

If we use $g(3^2) = 7/3$, we'd need another factor of $9/7 \approx 1.28$, but no odd prime gives a factor less than 1.8. Thus, we only have three solutions.

The valid integers are $15, 16$, and 27. Their sum is $15 + 16 + 27 = \boxed{58}$.