INTEGRATED MATHEMATICS TOURNAMENT

Team Round Solutions 2026 IMT

1. *(Brandon Jia)*: For positive integers I, M, and T satisfying $I \leq M \leq T$, the following equation holds true:

$$IMT = 5(I + M + T)$$

Compute the sum of all possible values of T across all valid (I, M, T).

Answer: 128

Solution: We can divide the given equation by IMT to obtain:

$$1 = 5\left(\frac{1}{MT} + \frac{1}{IT} + \frac{1}{IM}\right)$$

Since $I \leq M \leq T$, the largest fraction is $\frac{1}{IM}$, meaning $1 \leq \frac{15}{I^2}$, which gives $I^2 \leq 15$. The possible values for I are 1, 2, and 3.

For $I = 1$, the equation becomes $MT = 5(1+M+T)$. Rearranging and adding 30 to both sides gives $(M-5)(T-5) = 30$. Since $M \leq T$, the valid pairs for (M, T) are $(6, 35), (7, 20), (8, 15)$, and $(10, 11)$.

For $I = 2$, the equation becomes $2MT = 5(2 + M + T)$. Rearranging and factoring gives $(2M - 5)(2T - 5) = 45$. The valid pairs for (M, T) are $(3, 25), (4, 10)$, and $(5, 7)$.

For $I = 3$, the equation is $3MT = 5(3 + M + T)$, which factors as $(3M - 5)(3T - 5) = 70$. The valid pair with integers is $M = 4, T = 5$.

The possible values for T are 35, 20, 15, 11, 25, 10, 7, and 5. Their sum is $\boxed{128}$.

2. *(Brandon Jia)*: A polynomial of degree 6 is defined as

$$P(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

The six coefficients a_0, a_1, a_2, a_3, a_4, and a_5 are chosen independently and uniformly at random from the set $\{1, 2, 3, 4, 5, 6\}$ by rolling six standard fair dice. Let $r_1, r_2, r_3, r_4, r_5, r_6$ be the six complex roots of $P(x)$. The expected value of the sum of the squares of the roots, $r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2$, can be expressed as a fraction $\frac{m}{n}$ in lowest terms. Compute $m + n$.

Answer: 55

Solution: Using Vieta's formulas, the sum of the squares of the roots is:

$$r_1^2 + r_2^2 + \cdots + r_6^2 = (\sum r_i)^2 - 2\sum_{i<j} r_ir_j = (-a_5)^2 - 2a_4 = a_5^2 - 2a_4$$

Because the expected value is linear, we can separate this into $E[a_5^2] - 2E[a_4]$. For a standard fair die, the expected value of a single roll is $E[a_4] = 3.5 = 7/2$. The expected value of its square is $E[a_5^2] = (1 + 4 + 9 + 16 + 25 + 36)/6 = 91/6$. Substituting these in, we find the expected sum is $91/6 - 2(7/2) = 49/6$. Thus, $m = 49$ and $n = 6$, giving $m + n = \boxed{55}$.

3. *(Brandon Jia)*: Brandon and Christopher play a game starting with the number 2026^{2026} on a chalkboard. On their turn, a player can subtract any proper divisor of the current number to create a new number, which replaces the old one. The player who is forced to leave the number 1 on the board loses. Assuming optimal play, the first player can always win. Compute the number of distinct integers the first player can subtract on their first turn to guarantee a win.

   Answer: 2027

   Solution: The starting number $N = 2026^{2026}$ can be factored as $2^{2026} \cdot 1013^{2026}$, which is an even number. If a player has an even number, they can always subtract an odd divisor to leave an odd number for their opponent. If a player is given an odd number, all of its divisors are odd. Subtracting an odd divisor from an odd number always leaves an even number.

   Because the game ends when a player is forced to leave 1 (an odd number), the player who can consistently pass an odd number to their opponent will win. Since the first player starts with an even number, they must subtract an odd proper divisor to pass an odd number. The odd divisors of N are the divisors of 1013^{2026}, which are $1013^0, 1013^1, \ldots, 1013^{2026}$.

   There are $\boxed{2027}$ such odd divisors. All of them are smaller than N, so they are all valid first moves to guarantee a win.

4. *(Christopher Liang)*: In isosceles triangle IMT with $IT = MT$ and $IM = 2026$, let the midpoint of IM be A and MT be B. The angle bisector of I bisects segment AB. The value of $\cos(T)$ can be written as $\frac{m}{n}$ for relatively prime integers m, n. Find $m + n$.

Answer: 15

Solution: Let $c = IM$ and $a = IT = MT$. Any point on the angle bisector of $\angle I$ is equidistant from the sides IM and IT. Let D be the intersection of the angle bisector with segment AB. Since the angle bisector bisects AB, D is the midpoint of AB. This means D's distance to IM and its distance to IT must be equal.

We compute the distance from D to IM. Since D is the midpoint of AB, its distance to IM is the average of the distances from A and B to IM. Point A lies on IM, so its distance to IM is 0. Let h be the altitude from T to IM. Because B is the midpoint of MT, its distance to IM is $h/2$. Thus, the distance from D to IM is the average of 0 and $h/2$, which is $h/4$.

Next, we compute the distance from D to IT. Similarly, this is the average of the distances from A and B to IT. Let h_I be the altitude from M to IT. By equating the area of the triangle calculated from both bases, we know $\frac{1}{2}ch = \frac{1}{2}ah_I$, which gives $h_I = \frac{ch}{a}$.

Point B is the midpoint of MT, so its distance to IT is half the altitude from M, or $h_I/2$. Point A is the midpoint of IM, and since I is on IT, A's distance to IT is also half the altitude from M, which is $h_I/2$. The average of these two equal distances is $h_I/2$.

Substituting $h_I = \frac{ch}{a}$ into this expression gives the distance from D to IT as $\frac{ch}{2a}$.

Equating the distances from D to the two sides gives:

$$\frac{h}{4} = \frac{ch}{2a}$$

Dividing both sides by h leaves $1/4 = c/(2a)$, meaning $a = 2c$. The legs of the isosceles triangle are twice the length of the base.

To find $\cos(T)$, we apply the Law of Cosines to $\triangle IMT$:

$$c^2 = a^2 + a^2 - 2a^2\cos(T)$$

Substituting $a = 2c$ gives:

$$c^2 = 8c^2 - 8c^2\cos(T)$$

Dividing by c^2 and rearranging yields $8\cos(T) = 7$, so $\cos(T) = 7/8$.

Thus, $m = 7$ and $n = 8$, making $m + n = \boxed{15}$.

5. *(Brandon Jia)*: Compute the total number of digits of the smallest positive integer N that satisfies:

   - N is a multiple of 2026.
   - N ends with the digits "2026".
   - The sum of the digits of N is 2026.

   Answer: 229

   Solution: Since N ends in the digits 2026, we can express it as $N = 10000k + 2026$ for some integer $k \geq 0$. We want N to be a multiple of 2026, so $10000k + 2026 \equiv 0 \pmod{2026}$, which simplifies to $10000k \equiv 0 \pmod{2026}$.

   Dividing by 2 gives $5000k \equiv 0 \pmod{1013}$. Because 1013 is prime, k must be a multiple of 1013. To minimize the number of digits of N, we want k to have the fewest digits possible while making the total digit sum of N equal to 2026.

   The digits "2026" sum to 10, so the digits of k must sum to 2016. The minimum number of digits to achieve a sum of 2016 is $\lceil 2016/9 \rceil = 224$. This would require k to be written as 224 nines, or $10^{224} - 1$.

   However, $10^{224} - 1$ is not divisible by 1013. We can deduce this because 1013 is prime, so the order of 10 modulo 1013 must divide 1012. Since 224 does not divide 1012, $10^{224} - 1$ is not a multiple of 1013.

   Thus, k must have at least 225 digits. A number with 225 digits provides enough degrees of freedom to find a multiple of 1013 whose digits sum to 2016. Appending the 4 digits from "2026" to k means N will have $225 + 4 = \boxed{229}$ digits.

6. *(Christopher Liang)*: Three positive real numbers, x, y, and z, are chosen at random from the interval $(0, \pi)$, and satisfy the condition $x + y + z = \pi$. A triangle T is constructed with side lengths of $\sin(x)$, $\sin(y)$, and $\sin(z)$. The probability that T is stricly acute can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

   Answer: 5

   Solution: Because x, y, and z are positive and sum to π, they can represent the three angles of a valid triangle. By the Law of Sines, a triangle with angles x, y, and z has side lengths that are proportional to $\sin(x), \sin(y)$, and $\sin(z)$. This means the constructed triangle T is similar to a triangle with angles x, y, and z.

   For T to be strictly acute, all three of its angles must be less than $\pi/2$. This requires $x < \pi/2$, $y < \pi/2$, and $z < \pi/2$. The sample space of all possible combinations (x, y, z) forms an equilateral triangle. The sub-region where all three angles are less than $\pi/2$ corresponds to the medial triangle formed by connecting the midpoints of the equilateral triangle.

   The area of the medial triangle is $1/4$ of the total area, making the probability $1/4$. Thus, $m = 1$ and $n = 4$, yielding $m + n = \boxed{5}$.

7. *(Brandon Jia)*: Determine the number of ways to fill all 25 squares of a 5×5 grid with either a 1 or a -1 such that the product of the numbers in every row is -1, and the product of the numbers in every column is -1.

Answer: 65536

Solution: We can independently choose the values of the top-left 4×4 subgrid. Since each of these 16 cells can be filled with either 1 or -1, there are 2^{16} ways to fill this portion.

For the first 4 rows, the value of the cell in the 5th column is uniquely determined because the product of the entire row must be -1. Similarly, for the first 4 columns, the value of the cell in the 5th row is uniquely determined to make the column product -1.

We must ensure the cell in the 5th row and 5th column is consistent. Evaluating it by its row requires $A_{5,5} = -1/\prod_{j=1}^{4} A_{5,j}$, and evaluating by its column requires $A_{5,5} = -1/\prod_{i=1}^{4} A_{i,5}$. Both expressions evaluate to $1/P$, where P is the product of all 16 cells in the 4×4 subgrid. Therefore, $A_{5,5}$ is always consistent.

Any of the 2^{16} configurations for the 4×4 subgrid works, giving us $\boxed{65536}$ ways.

8. *(Christopher Liang)*: Triangle ABC has $\angle B = 100^\circ$ and $AB = 1$. Let ω_1 be the circumcircle of $\triangle ABC$ with center O. A chord DE of ω_1, distinct from AC, is drawn such that $DE \parallel AC$ and $DE = AC$. Let ω_2 be a smaller circle centered at O and tangent to both BD and DE. If F and G are the points of tangency of ω_2 with BD and DE, respectively, determine the degree measure of the central angle $\angle FOG$.

Answer: 160

Solution: Circle ω_2 is tangent to BD and DE, meaning the center O is equidistant from these two chords. This implies the chords have the same length, so $BD = DE$. We are given $DE = AC$, meaning $AC = BD = DE$.

Because inscribed $\angle B = 100^\circ$, the minor arc AC measures $360^\circ - 2(100^\circ) = 160^\circ$. Since AC, BD, and DE are equal chords, they all subtend 160° minor arcs.

With $DE \parallel AC$ and equal in length, $ACED$ is a rectangle centered at O, meaning A and D are diametrically opposed, as are C and E. Setting A to 0°, we have D at 180° and E at 340°. For arc BD to be 160° with B on the minor arc AC, B must be placed at 20°.

The inscribed angle $\angle BDE$ subtends the arc from E (340°) to B (20°) not containing D, which spans 40°. Thus $\angle BDE = 20^\circ$. The quadrilateral $OFDG$ has right angles at the points of tangency F and G. Therefore, the central angle is $\angle FOG = 180^\circ - \angle BDE = \boxed{160^\circ}$.

9. *(Brandon Jia)*: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that for all positive real numbers x and y, the following equation holds:

$$f(xy) = f(x)^y \cdot f(y)^x$$

Given that $f(2) = 256$, find $\sqrt{f(3)}$.

Answer: 729

Solution: We can rewrite the function as $f(x) = e^{g(x)}$. Substituting this into the given equation yields:

$$e^{g(xy)} = e^{yg(x)}e^{xg(y)} = e^{yg(x)+xg(y)}$$

Equating the exponents gives $g(xy) = yg(x) + xg(y)$. Dividing the entire equation by xy results in:

$$\frac{g(xy)}{xy} = \frac{g(x)}{x} + \frac{g(y)}{y}$$

This shows that the function $\frac{g(x)}{x}$ has the property of a logarithm, meaning $\frac{g(x)}{x} = c\ln(x)$ for some constant c. This gives $g(x) = cx\ln(x)$, which translates back to $f(x) = e^{cx\ln(x)} = x^{cx}$.

We know $f(2) = 256$, so $2^{2c} = 256$, which means $2c = 8$ and $c = 4$. The function is $f(x) = x^{4x}$. We want to find $\sqrt{f(3)}$. First, $f(3) = 3^{12}$. Taking the square root gives $3^6 = \boxed{729}$.

10. *(Brandon Jia)*: The following system of equations has exactly k distinct real solution pairs (x, y).

$$x^5 - 10x^3y^2 + 5xy^4 = \frac{117x - 44y}{x^2 + y^2}$$

$$5x^4y - 10x^2y^3 + y^5 = \frac{44x + 117y}{x^2 + y^2}$$

Let these distinct solution pairs be denoted as $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$. Compute

$$\sum_{j=1}^{k}(x_j^2 + y_j^2)^3.$$

Answer: 500

Solution: Let $z = x + iy$. If we expand $z^5 = (x + iy)^5$, we can see that the left sides of the given equations are the real and imaginary parts of z^5. The right side of the equations can be combined as:

$$\frac{117x - 44y}{x^2 + y^2} + i\frac{44x + 117y}{x^2 + y^2} = \frac{(117 + 44i)(x - iy)}{|z|^2} = \frac{117 + 44i}{z}$$

This allows us to condense the system into a single complex equation: $z^5 = \frac{117+44i}{\bar{z}}$, which rewrites as $z^4|z|^2 = 117 + 44i$.

Taking the magnitude of both sides gives $|z|^6 = |117 + 44i|$. We evaluate the magnitude: $\sqrt{117^2 + 44^2} = \sqrt{13689 + 1936} = \sqrt{15625} = 125$. So $|z|^6 = 125$, meaning $|z|^2 = 5$.

Substituting this back gives $5z^4 = 117 + 44i$. This complex number equation has 4 distinct roots for z, corresponding to 4 distinct real solution pairs (x, y), meaning $k = 4$. For each solution, $x^2 + y^2 = |z|^2 = 5$. The required sum is $\sum_{j=1}^{4}(x_j^2 + y_j^2)^3 = 4 \times 5^3 = \boxed{500}$.